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PROJECT COMPLETION
REPORT NO. 451X

Integrated System Identification and Optimization for Conjunctive Use of Ground and Surface Water

Phase II

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Office of
Water Research and Technology
United States Department
of the Interior

CONTRACT NO.
B-050-OHIO

INTEGRATED SYSTEM IDENTIFICATION AND OPTIMIZATION FOR CONJUNCTIVE USE OF GROUND AND SURFACE WATER

Phase II

Submitted to The

Office of Water Research and Technology

U.S. Department of the Interior

Washington, D.C.

and

State of Ohio

Water Resources Center

Ohio State University

Contract No. B-050-OHIO

by

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Report No. SED-WRG-74-1

September, 1974

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ACKNOWLEDGEMENTS

We would like to express our gratitude to the following organizations and individuals who have been invaluable in our research efforts in this project. First to the Office of Water Research and Technology (formerly Office of Water Resources Research), U.S. Department of the Interior. Particularly to its acting director, Dr. W.A. Hall and to the project officer, Dr. J. Velatis, for their support of this study. To the Ohio State University, Water Resources Center, and in particular to Dr. K.S. Shumate, former Director of the Center, Dr. Robert C. Stiefel, Director, and to Mr. James F. Teeple, Assistant to the Director, for their assistance and advice. To the Miami Conservancy District, Dayton, Ohio, and specifically to Messrs. Robert Schroer, Chief Engineer, and Paul Plumer who provided data and significant advice and support. To the staff of the U.S. Geological Survey, Water Resources Division in Washington, D.C., and particularly to Drs. Nicholas C. Matalas, Chief of Systems Engineering Group, and Thomas Maddock, III, who provided a documented aquifer simulation model. The collaboration and continuous exchange of information with the staff of the Water Resources Division, U.S.G.S. have significantly enhanced the progress of this project. We would like to thank the faculty members at Case Western Reserve University who served on the thesis committees of the graduate students involved in this project, in particular to Professors G.L. Blankenship and D. Macko. Finally, we would like to thank Mr. Dave Nicol for his review and editorial comments.

EXECUTIVE SUMMARY

Highlights of Phase I Project Efforts

The following four points indicate the direction and accomplishments of the first year of project activity.

1. In a continuing effort to improve the knowledge of the aquifer system characteristics, a computer program which simulates an aquifer using an alternating direction implicit-iterative procedure has been obtained from the U. S. Geological Survey. This aquifer simulation model was utilized as a component in the system identification effort.

2. The search for a real data base for the final implementation and validation of the models has been very successful. We have established a working relationship with the Miami Conservancy District and their staff have expressed its interest in our research and promised to collaborate and assist in supplying groundwater and other hydrological data from the District. This has aided in identifying the model parameters and in validating and testing the model once the identification was completed. Furthermore, water quality models currently under development by the District engineers should be useful when the validated models are used to derive improved water management policies.

3. Our efforts have focussed on the development of improved solution methodologies for the identification and optimization problems. A system identification methodology has been developed applicable to calibration of confined and unconfined (under certain constraints) aquifer models described by parabolic partial differential equations. Work has been completed on the formulation and coding of a digital computer

software package capable of estimating the values of transmissivity, as a function of location within the aquifer. The estimation is based on measured and observed data on the waterhead and the various flows from and into the aquifer. The identification methodology has been successfully applied to the parameter identification of an aquifer model simulating the behavior of a real aquifer system. The Fairfield-New Baltimore aquifer in southern Ohio was chosen as the problem site in collaboration with the Miami Conservancy District. After several meetings with personnel from the Miami Conservancy District and after the initial stage of data collection was completed, a two dimensional partial differential equation model was postulated for the region. Calibration of the model was pursued using the new system identification methodology developed in this research. The results are satisfactory and establish that analytical means, as opposed to simulation by trial-and-error techniques, are feasible and yield excellent results. The identification phase has been completed and is integrated with the overall ground and surface management model in Phase II.

4. The product of Phase I is a complete, validated aquifer model which we believe is most valuable to engineers and managers concerned with groundwater systems. It can be used both for simulation and/or optimization. In particular it is of direct value to engineers and analysts wishing to know the response of the aquifer system to various demands placed upon it. The model was fully utilized, of course, for the conjunctive management of ground and surface water developed in Phase II of this project. The model is in the form of a FORTRAN V computer program.

Highlights of Phase II

The following nine points indicate the accomplishments of Phase II (the second year) of project activity.

1. The groundwater parameter identification model developed in Phase I has been further modified and improved as part of Phase II work. The modified algorithm not only shows more accuracy in results but also it is superior in computational feasibility, primarily because of fast convergence of the algorithm.

2. The Fairfield-New Baltimore aquifer system in the Great Miami River Valley which was used as a real data base for the identification and validation of the model developed in Phase I was also used for Phase II work. By using the same data base, a direct comparison with the results of Phase I was available. The transmissivity values identified in Phase II provide a closer prediction of drawdown response than the results of Phase I.

3. As a contribution to water resources modeling techniques a procedure for simulation of a complex groundwater system was developed utilizing the hierarchical modeling approach. The technique of using a higher level model to provide boundary conditions for an isolated aquifer cell proved to be quite effective.

4. The coupling of a complex multicell aquifer system with a general management model was successfully achieved and tested. This extension of the algegraic technological function approach was basically developed by application of the superposition technique.

5. Functions were developed to relate the infiltration from a stream into an aquifer due to pumpage at a well. The general system is represented by a stream network traversing a multi-

cell aquifer system. Changes in infiltration rates in a particular stream due to pumpage were divided into two sources; (i) that from pumpage in the close vicinity of the stream, and (ii) that from pumpage relatively farther away. By superimposing the system response, a differential weighting was used to aggregate well pumpage in the derivation of stream aquifer interactions.

6. A new approach was proposed to incorporate time varying boundary conditions in a ground water system. By using techniques already developed, a system subject to such conditions can be coupled with any desired management model.

7. A predictive model was developed to demonstrate the usefulness of these response oriented models. The basic formulation was achieved by superposition of the system response. The decision variables included pumpage, artificial recharge, and imported water.

8. A model for the planning and management of a groundwater system was developed. This model incorporates a new approach to a tax-quota system. A modification of this model, using developments made available in Phase II study provides a management procedure for the optimal conjunctive use of ground and surface water.

9. Some of the research results were applied to the area of the lower Great Miami River with data provided by the Miami Conservancy District (MCD), Dayton, Ohio. The modeling procedure shows a considerable advantage over previous models in application to the Fairfield area in this valley. In fact, the MCD is particularly interested in applying the procedure to model the Dayton area. This would allow the district to use their own computer (32K) directly, rather than having to use the conventional model of the U.S.G.S. which requires a large digital computer.

The remainder of the research results were also tested on the data for this Fairfield case study, again demonstrating a real advantage in comparison to any other available methodologies.

The main contribution of this research arise in the formulation and solution of management models which include the responsive aspects of more complex water resources systems. In this regard the MCD representatives are very interested, and a widespread utilization of these results by other agencies and institutions can be anticipated.

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A QUADRATIC GROUNDWATER IDENTIFICATION MODEL: AN EXTENSION TO PHASE I

1.1 Introduction

The identification model for groundwater parameters and the solution methodology developed in Phase I of this research project has been further modified and improved as part of Phase II. The modification basically results from further consideration of the distributed nature of transmissivity values over the aquifer system. This has been achieved by representing the mathematical structure of the transmissivity function as a second order polynomial of spatial coordinates, distributed over the aquifer system. The identification algorithm developed in Phase I was modified to include a convergence criteria and convergence is now much faster. This modification is important for maintaining computational feasibility and tractability since the complexity of the identification problem is vastly increased due to the second order polynomial approximation of transmissivity function. The Fairfield-New Baltimore aquifer system in the Great Miami River Valley which was used as a real data base for the identification and validation of the model developed in Phase I was also used for Phase II. Thus we are able to compare results of the current work directly with the results of Phase I.

It is important to appreciate the reasons for incorporating the more complex modeling procedures used in Phase II. A considerable portion of Section 2.3 and 2.4 within Chapter II of the report on Phase I (Haimes, 1973) was devoted to a discussion on alternative aquifer models, particularly their advantages and disadvantages from the standpoint of computational ease and with regard to physical representation. One fact is clear. The computationally simplest representation--constant parameters representing a relatively large (spatially) region--proves

to be of limited usefulness. The deposition of aquifer material occurs over very long time periods through a sequence of natural processes. Often a good aquifer results when material with good storage capacity and an unusually high transmissivity happens to be suitably located. In regions outsides the "best" locations, however, the transmissivity of the aquifer material is certain to taper off to somewhat lower levels. Thus, while representation by constant parameters may sometimes provide a satisfactory matching of response, it is likely that in general a much better result is available by recognizing the non-constant, distributed nature of aquifer properties. The representation used here, with quadratically variable properties, can represent both linear trends (in any direction) and ellipsoidal variation (along appropriate axes) as well as the superposition of such features. The approach seems sufficiently complex to serve any foreseeable needs.

1.2 Statement of the Groundwater Model and Identification Problem:

The following second order parial differential equation [Jacob, 1950] is used to represent the groundwater system of the Fairfield-New Baltimore area:

$$\frac{\partial}{\partial x}\left[T(x,y)\frac{\partial h}{\partial x}\right] + \frac{\partial}{\partial y}\left[T(x,y)\frac{\partial h}{\partial y}\right] = S(x,y)\frac{\partial h}{\partial t} + Q(x,y,t) \qquad (1.1)$$

where

$T(x,y)$ = Non-homogeneous transmissivity (ft/sec)
$h(x,y,t)$ = Hydraulic head (ft)
$S(x,y)$ = Storage coefficient
$Q(x,y,t)$ = Net pumping rate per unit area, including recharge, leakage etc. (ft/sec)

The computational procedure for solving equation (1.1) involves a finite difference approximation, using an alternate direction implicit iterative scheme. This procedure was documented at the end of Phase I [Haimes, 1973]. Also, the details of the identification problem [Lopez, 1973 and Lions, 1971], and the computational algorithm [Marquardt, 1963]

are well documented in the earlier report [Haimes, 1973]. These previous developments will be repeated here only as necessary for continuity in the presentation of the modified identification scheme.

To solve the groundwater system model of equation (1.1), boundary and initial conditions must be added:

$$\frac{\partial}{\partial x}\left(T(x,y)\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T(x,y)\frac{\partial h}{\partial y}\right) = S(x,y)\frac{\partial h}{\partial t} + Q(x,y,t) \qquad (1.2)$$

$$\mathbf{h}(x,y,o) = h_o \qquad (1.3)$$

$$\frac{\partial h}{\partial \vec{n}}\Big|_{r_1} = \phi, \quad h(x,y,t)\Big|_{r_2} = h_1 \qquad (1.4)$$

and $Q(x,y,t) \in R$.

where (1.3) is the initial condition of the system, (1.4) the boundary conditions (which may be constant flow and/or constant head), r_1 and r_2 denote the boundary geometry, and R is the domain of (1.2) - (1.4) defined as $\Omega \times [0,T]$, where Ω is the area of the aquifer, and t represents time.

Since the transmissivity function $T(x,y)$ is unknown, the response $h(x,y,t)$, cannot be computed from the (1.2), (1.3), and (1.4). All other parameters such as pumpage, initial and boundary conditions, storage coefficients, etc. are assumed known, as functions of space and time. The identification problem is to estimate the value of the transmissivity function $T(x,y)$, such that a specified performance criterion is satisfied. This choice of a performance function, however, depends on many factors, including, for example, the model representing the physical system, the number of data points, the sensitivity of parameters as related to performance function, and so on.

A least square norm of the output error (i.e. between observed and calculated values for the water head) was selected as the loss (or performance) function. This function J can be expressed as:

$$J(T(x,y)) = \int_0^t \int_\Omega \left[h(x,y,t;T) - \hat{h}(x,y,t)\right]^2 dt\, d\Omega \qquad (1.5)$$

where, $h(x,y,t;T)$ is the response of the model (1.2)-(1.4) for a given transmissivity function $T(x,y)$, and $\hat{h}(x,y,t)$ is the observed value of the water head at various points in space and time over the areal distribution of the aquifer. The optimal identification problem can now be stated as follows:

$$\text{Minimize } J(T(x,y)) = \text{Min} \left\{ \int_0^t \int_\Omega [h(x,y,t;T) - h(x,y,t)]^2 dt\, d\Omega \right. \qquad (1.6)$$

subject to the constraints set

$$\left.\begin{aligned} &\frac{\partial}{\partial x}\left(T(x,y)\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T(x,y)\frac{\partial h}{\partial y}\right) = S(x,y)\frac{\partial h}{\partial t} + Q(x,y,t) \\ &h(x,y,o) = h_o \\ &\frac{\partial h}{\partial \vec{n}}\bigg|_{r_1} = \phi;\ h(x,y,t)\bigg|_{r_2} = h_1 \end{aligned}\right\} \qquad (I)$$

and

$$Q(x,y,t)\ \varepsilon R$$

$$T(x,y)\ \varepsilon (E^1)^+$$

The search for a transmissivity function $T(x,y)$ to minimize the objective function (1.6) subject to the constraints set (I) constitutes the identification algorithm.

1.3 The Identification Algorithm as a Static Optimization Problem

Since, in general, it is not possible to observe the drawdown for every point in the aquifer as a continuous function of time, the continuous objective functional (1.5) is replaced by a discretized approximation:

$$J(T(x,y)) \simeq \phi(T(x,y)) = \sum_{i=1}^{N} \left[h(\underline{x}_i;T) - \hat{h}(\underline{x}_i) \right]^2 \qquad (1.7)$$

where $\underline{x} = (x_1,x_2,x_3)$

$= (x,y,t)$

and $i = 1,2,\ldots N$, denotes the i^{th} observation point and N represents the total number of output observation points.

To extend the results of Phase I, a general second order polynomial representation of the transmissivity function $T(x,y)$ is proposed. Of course, the higher the degree of such a polynomial, the more complex the identification problem becomes computationally. This second order polynomial, in spatial coordinates over the aquifer, can be represented as follows:

$$T(x,y) = b_1x^2 + b_2 y^2 + b_3 x + b_4 y + b_5 \qquad (1.8)$$

where b_1, b_2, b_3, b_4, b_5, are the unknown coefficients which must be estimated.

The identification algroithm is based on a maximum neighborhood method. This was developed by [Marquardt, 1963] and is often referred to as the Marquardt algorithm. In effect, the algorithm performs an optimum interpolation between the Taylor Series and gradient methods, based upon the maximum neighborhood in which a truncated Taylor Series gives an adequate representation of the nonlinear model. The details of the mathematical development of the Marquardt algorithm was documented

previously (Haimes [1973]), and will not be discussed in detail here.

The simulation response of (1.1) as a function of the unknown transmissivity can be compactly expressed as:

$$h(x,y,t;T) = F(x_1,x_2,x_3,b_1,b_2,b_3,b_4,b_5) \tag{1.9}$$

Similarly, the objective function can be expressed in terms of the unknown parameters b_i, $i = 1, 2, \ldots, 5$:

$$\phi(\underline{b}) = \sum_{i=1}^{N} \left[F(\underline{x}_i,\underline{b}) - \hat{F}(\underline{x}_i)\right]^2 \tag{1.10}$$

Expanding equation (1.9) in a Taylor series about an initial estimate of the parameter vector $\underline{b}$, and truncating beyond linear terms, yields

$$\begin{aligned} h_\ell(\underline{x}_i,\underline{b}^\circ + \underline{\delta}_t) &= F(\underline{x}_i,\underline{b}^\circ) + \nabla F^T(\underline{X}_i,\underline{b}^\circ)\underline{\delta}_t \\ &= F(\underline{X}_i,\underline{b}^\circ) + \sum_{j=1}^{K} \left[\frac{\partial F}{\partial b_j}(\underline{X}_i,\underline{b})\bigg|_{\underline{b}=\underline{b}^\circ}\right]\delta_{tj} \end{aligned} \tag{1.11}$$

where, $\underline{\delta}_t$ = correction vector to $\underline{b}$
h_ℓ = linearized aquifer model output
K = population values of parameters

Substituting this linearized model output into objective function (1.10), yields

$$\phi(\underline{b}) \simeq \sum_{i=1}^{N}\left[F(\underline{X}_i,\underline{b}_o) + \sum_{j=1}^{K} \frac{\partial F}{\partial b_j}(\underline{X}_i,\underline{b})\bigg|_{b=b^\circ} \delta_{tj} - \hat{F}(\underline{X}_i)\right]^2 \tag{1.12}$$

Equation (1.12) gives the value of the objective function, $\phi(\underline{b})$ predicted by a linear estimate of $h(\underline{x}_i,\underline{b})$, given the state observations $\hat{h}(\underline{X}_i)$. The correction vector $\underline{\delta}_t$ appears linearly in (1.11), and thus the optimal value $\underline{\delta}_t^*$ can be determined by standard least-squares method, setting $\frac{\partial\phi(\underline{b})}{\partial\delta} = 0$ for all j. This yields

$$[A]\underline{\delta}_t = \underline{g} \tag{1.13}$$

where $[A] = [P]^T[P]$

$$[P] = \begin{bmatrix} \frac{\partial F_1}{\partial b_1} & \frac{\partial F_1}{\partial b_2} & \frac{\partial F_1}{\partial b_3} & \frac{\partial F_1}{\partial b_4} & \frac{\partial F_1}{\partial b_5} \\ \frac{\partial F_2}{\partial b_1} & \frac{\partial F_2}{\partial b_2} & \frac{\partial F_2}{\partial b_3} & \frac{\partial F_2}{\partial b_4} & \frac{\partial F_2}{\partial b_5} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F_N}{\partial b_1} & \frac{\partial F_N}{\partial b_2} & \frac{\partial F_N}{\partial b_3} & \frac{\partial F_N}{\partial b_4} & \frac{\partial F_N}{\partial b_5} \end{bmatrix}_{\underline{b} = \underline{b}^o}$$

The gradient vector $\underline{g} = (g_1, g_2, g_3, g_4, g_5)^T$

and $g_j = \sum_{i=1}^{N} \left[\hat{F}_i - F_i\right] \frac{\partial F_i}{\partial b_j}$

$j = 1, 2, 3, 4, 5.$

The algorithm which evaluates the aquifer response $h(x,y,t)$, the sensitivity functions $\frac{\partial F_i}{\partial b_j}$ for all i and j, the gradient vector $\underline{g}$, and the other variables, is essentially that discussed in Haimes [1973]. A few modifications and extensions were necessary to incorporate the five unknown parameters b_j, $i = 1, 2, \ldots 5$, and the polynomial representation of the transmissivity function $T(x,y)$.

1.3.1. Convergence Criteria in Optimal Identification: An Iterative Scheme.

It has been noted earlier that the task of an identification algorithm is to systematically search for those parameter values which minimize some performance function. An iterative gradient algorithm is developed here, based on the maximum neighborhood method [Marquardt, 1963]. In this algorithm, the direction and step size of the correction vector $\underline{\delta}_t$ are determined simultaneously by performing an optimal interpolation between the Taylor series and steepest descent methods. Marquardt proposes a modification of equation (1.13) to satisfy a maximum neighborhood principle:

$$(A + \lambda I)\underline{\delta}_t = \underline{g} \tag{1.14}$$

The value of λ, defined as the Marquardt parameter, orients the correction vector $\underline{\delta}_t$ between the directions indicated by the steepest descent and Taylor series methods. A critical factor in achieving fast convergence is the proper choice of the value of λ, which reduces the objective function $\phi(\underline{b})$ of equation (1.12) at each iteration. Marquardt [1963] has proved that there exists some λ which assures minimization of (1.12) for each iteration. By choosing the "proper" λ, the following condition is assured;

$$\phi(\underline{b}^{(r+1)}) < \phi(\underline{b}^{(r)}) \tag{1.15}$$

where, the superscript (r) denotes the iteration number.

1.3.2 Construction of the Convergence Algorithm:

Consider equation (1.14), denoting the iteration number by the superscript r:

$$(A^{(r)} + \lambda^{(r)} I)\ \underline{\delta}_t^{(r)} = \underline{g}^{(r)} \tag{1.16}$$

Equation (1.16) is solved for $\underline{\delta}_t^{(r)}$, yielding a new trial vector of parameters $\underline{b}^{(r)}$;

$$\underline{b}^{(r+1)} = \underline{b}^{(r)} + \underline{\delta}_t^{(r)} \tag{1.17}$$

which leads to a new sum of squares $\phi^{(r+1)}(\underline{b})$. Some form of trial and error procedure is required to choose a value of $\lambda^{(r)}$ that satisfies (1.15), and yet produces a rapid convergence to the least square values.

A convergence strategy was developed which is superior in speed to that of our previous work, and thus is computationally much more attractive. Minimizing the computer time requirements of solving these large problems can be very important.

For each iteration (r), it is desired to minimize $\phi(b^{(r)})$ in the maximum neighborhood over which the linearized function adequately represents the nonlinear function of equation (1.7), J((T(x,y)). Therefore, it is preferable to choose a small value of $\lambda^{(r)}$ whenever conditions are such that the unmodified Taylor series method would converge quickly. Large values of $\lambda^{(r)}$ are chosen only when is necessary to satisfy the condition.

$$\phi(\underline{b}^{(r+1)}) < \phi\ (\underline{b}^{(r)})$$

Consequently the following strategy was adopted as the convergence algorithum.

1. Denote $\lambda^{(r-1)}$ as the value of λ from the previous iteration. A choice of $\lambda^{(o)} = 0.1$ was found to be a good starting value.

2. Compute $\phi(\lambda^{(r-1)})$ and $\phi(\lambda^{(r-1)}/\nu)$, where ν is a parameter, $\nu > 1$. Then perform the following:

   (i) If $\phi(\lambda^{(r-1)}/\nu) < \phi^{(r)}$, set $\lambda^{(r)} = \lambda^{(r-1)}/\nu$.

   (ii) If $\phi(\lambda^{(r-1)}/\nu) > \phi^{(r)}$ and $\phi(\lambda^{(r-1)}) \leq \phi^{(r)}$, do not change $\lambda^{(r)}$, i.e. set $\lambda^{(r)} = \lambda^{(r-1)}$

   (iii) If $\phi(\lambda^{r-1)}/\nu) > \phi^{(r)}$ and $\phi(\lambda^{(r-1)}) > \phi^{(r)}$, increase the value of $\lambda^{(r)}$ successively within the iteration according to the following scheme,

   $$\lambda^{(r)} = \lambda^{(r-1)}\nu^{w}$$

   until for some smallest w, the condition

   $$\phi(\lambda^{(r-1)}\nu^{w}) \leq \phi^{(r)}$$

   is satisfied. A choice of initial value of w = 1.5, with an increment of 1.0 at each cycle within the iteration, was found to be appropriate for this problem.

3. The identification parameters b_i, i = 1, 2, 3, 4, 5 is complete only when either of the following two conditions are satisfied:

   (a) $$\left| \frac{\delta_{tj}(r)}{\tau + |b_j(r)|} \right| \leq \varepsilon_1, \quad j = 1, 2, 3, 4, 5$$

   (b) $$\left| \phi(b^{(r+1)}) - \phi(b^{(r)}) \right| \leq \varepsilon_2$$

   for some suitable choice of ε_1, ε_2 (both greater than zero), and τ. The choice of $\varepsilon_1 = 10^{-5}$, $\varepsilon_2 = 1$, and $\tau = 10^{-3}$ was found to be reasonably good in practice.

Chapter 2

COMPUTATIONAL RESULTS FOR THE QUADRATIC MODEL

2.1 Introduction

The Identification Algorithm developed in previous chapter has been applied to the estimation of transmissivity values for the Fairfield - New Baltimore aquifer system. The model estimated parameters for transmissivity functions were then used for model validation to establish the capability of the model to predict real system behavior. This aquifer system was also used previously as a real data base for the identification and validation of the model developed in Phase I. This provides a direct comparison of the results of Phase II with those of Phase I.

Finally, a sensitivity analysis was performed for the variation of parameter values with errors in observed data, changes in storage coefficients, and so on.

2.2 Parameters Identification and Model Validation

2.2.1 Model Calibration

The calibration of the model described in equation (1.2) was performed for the Fairfield - New Baltimore aquifer system. The basic hydrological and physical data for this aquifer system was available from [Spieker, 1968,] and further information was provided by the Miami Conservancy District, Dayton, Ohio. The time period 1952 to 1962 was chosen for the identification and validation processes. The period 1952 - 1962 was divided in two subperiods. Date from 1952 to 1956 was used for model identification. Once the model was identified, the entire period from 1952-1962 was used in the model validation.

Table 2.1 summarizes the characteristics of the aquifer in the Fairfield - New Baltimore area. The area of the aquifer system modeled and the location of the pumping wells are shown in Figure 2.1. The constant head and recharging boundaries are indicated in Figure 2.2. The infiltration rates and the complete pumping history of the region from 1952 to 1962 are presented in Table 2.2 and Table 2.3 respectively. Observations of water head at 25 grid points for each year from 1952 to 1956, as used for the model identification are presented in Table 2.4.

Characteristic	Description
Aquifer	Unconfined. Small marginal areas are of semi-confined type
Storage Coeffiecient S	0.145 average (dimensionless)
Transmissivity Coefficient T	Unknown (ft^2/sec)
Initial Head	550 (ft) average water table height
Boundary Conditions	East & West bounds: Constant Head North & South bounds: 6.8 (mgd) flow on the average
Wells	Six pumping wells distributed in three areas: Southwestern Co. (13.8 mgd) Hamilton South (7.5 mgd) Atomic Energy Commission (1.0 mgd)
Approximate Area	30 square miles (a 20 x 30 node grid has been used to approximate this area)

TABLE 2.1

Aquifer Data: The Fairfield - New Baltimore Aquifer System

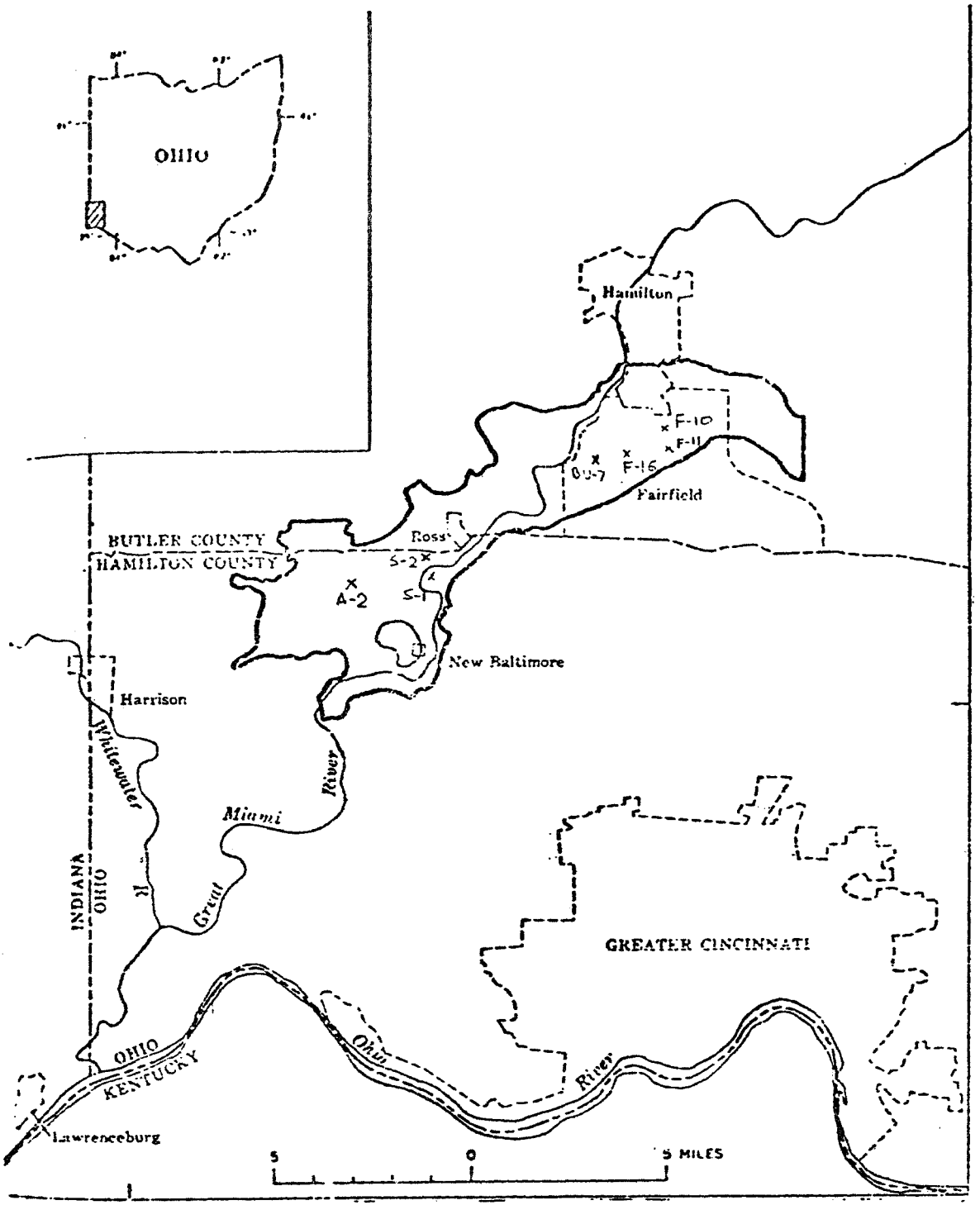

FIGURE 2.1

Location of the Fairfield - New Baltimore Aquifer System Modeled

Well Locations Marked (x)

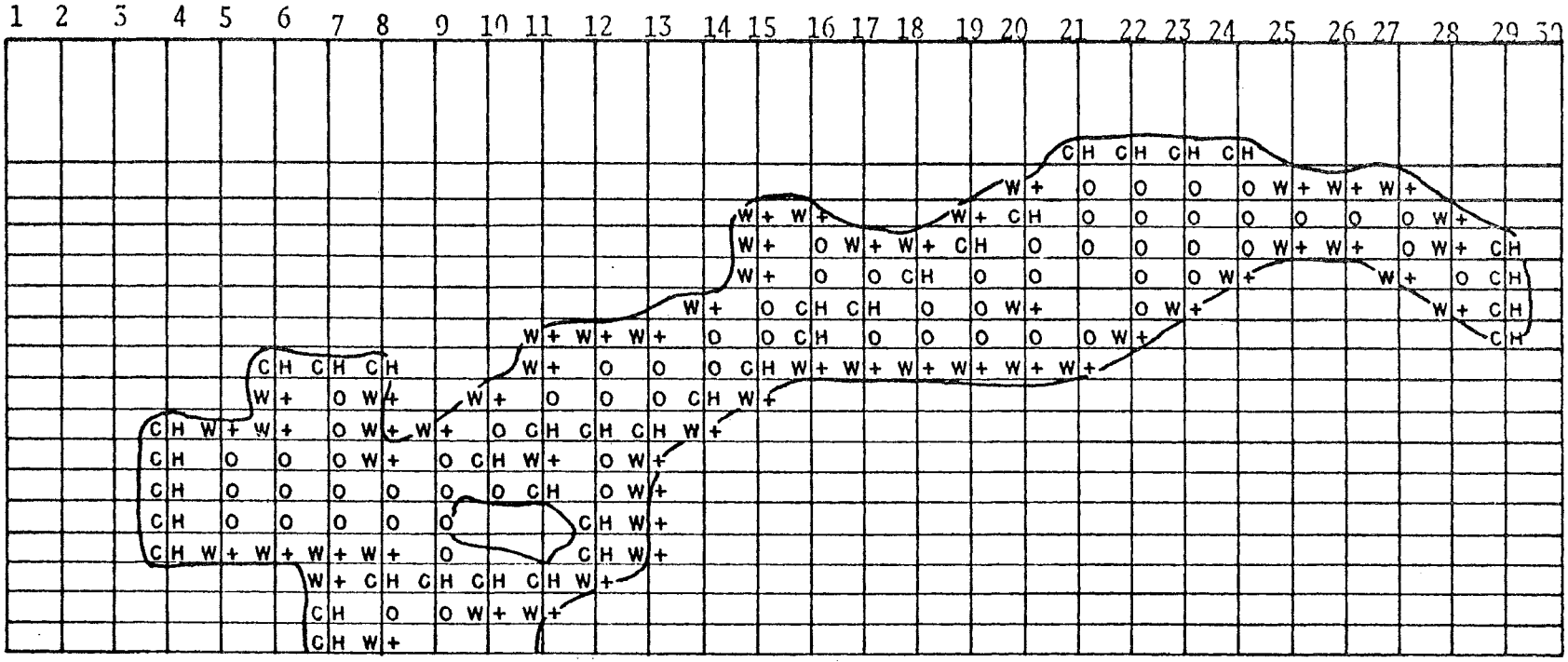

FIGURE 2.2

$T_{i-\frac{1}{2}}$

Computer Discretization of the Aquifer Area, CH Denotes Constant Head Boundaries, W+ Recharging Boundaries, and W- Pumping Wells.

Boundary Points (B.P.)	12,5	12,6	11,6	11,8	12,8	12,9	11,10	10,11	9,11	9,12	9,13	8,14
	(I,J) (See figure 5.5 for location of this coordinates)											
Infiltration Rate (I.R.)	7	5	5	7	3	3	10	8	5	5	6	8
B.P.	7,15	6,15	5,15	5,16	6,17	6,18	5,19	4,20	4,25	4,26	4,27	5,28	6,28
I.R.	8	5	8	6	5	5	8	9	6	5	5	7	3
B.P.	8,28	7,27	6,26	6,25	7,24	8,23	9,22	10,21	10,20	10,18	10,19	10,17	
I.R.	8	5	5	5	8	5	8	7	7	5	5	5	
B.P.	10,16	11,15	12,14	13,13	14,13	15,13	16,13	17,12	18,11	18,10	19,8		
I.R.	5	6	7	8	5	5	6	5	6	5	7		
B.P.	17,7	16,8	16,7	16,6	16,5								
I.R.	5	5	5	6	6								

TABLE 2.2

Infiltration Rates Fairfield-New Baltimore Aquifer. Units: $ft^3/sec.*100$

Well Coordinates	Well Name	PUMPING PERIODS										
		1952	1953	1954	1955	1956	1957	1958	1959	1960	1961	1962
13,8	A-2	155	155	155	155	155	155	155	155	155	155	155
13,11	S-1 S-2	1512	1835	1762	2155	2031	2260	2019	2298	2223	2004	1951
7,21	F-10	0000	0000	500	0000	338	377	381	372	356	354	357
8,21	F-11	0000	500	0000	0000	423	471	477	465	445	443	446
8,20	F-16	500	0000	0000	500	338	377	381	372	356	354	357

TABLE 2.3

Pumping history Fairfield-New Baltimore aquifer. Figures are given in ft.^{3}/Sec.x*100. Data from 1957-62 were not used in the identification of T.

Pumping Period = 1 (1952)			Pumping Period = 2 (1953)		
OBSERVATION POINT		DRAWDOWNS (ft)	OBSERVATION POINT		DRAWDOWNS (ft)
6	19	0	6	19	0
6	20	-0.511	6	20	-0.606
6	21	-0.611	6	21	-1.043
6	22	-0.534	6	22	-1.060
7	19	-0.745	7	19	-0.648
7	20	-1.459	7	20	-1.385
7	21	-1.187	7	21	-2.085
7	22	-0.879	7	22	-1.768
8	19	-1.514	8	19	-1.165
8	20	-3.459	8	20	-2.199
8	21	-1.853	8	21	-4.183
8	22	-1.219	8	22	-2.598
9	19	-1.311	9	19	-1.284
9	20	-1.950	9	20	-2.003
9	21	-1.633	9	21	-2.737
9	22	-1.337	9	22	-2.581
12	8	-0.503	12	8	-0.566
12	10	-0.041	12	10	-0.051
13	7	-0.556	13	7	-0.629
13	8	-1.338	13	8	-0.425
13	9	-0.462	13	9	-0.513
13	11	-6.575	13	11	-8.086
13	12	-2.071	13	12	-2.603
14	8	-0.566	14	8	-0.641
14	12	-0.726	14	12	-0.942

TABLE 2.4

Water Head Observations, Fairfield - New Baltimore Aquifer System

Pumping Period = 3 (1954)			Pumping Period = 4 (1955)		
OBSERVATION POINT		DRAWDOWN (ft)	OBSERVATION POINT		DRAWDOWN (ft)
6	19	0	6	19	0
6	20	-0.840	6	20	-0.646
6	21	-1.669	6	21	-0.861
6	22	-1.436	6	22	-0.831
7	19	-0.656	7	19	-0.871
7	20	-1.702	7	20	-1.715
7	21	-3.751	7	21	-1.558
7	22	-2.187	7	22	-1.278
8	19	-0.887	8	19	-1.728
8	20	-1.575	8	20	-3.788
8	21	-2.304	8	21	-2.275
8	22	-2.033	8	22	-1.676
9	19	-0.917	9	19	-1.576
9	20	-1.380	9	20	-2.314
9	21	-1.767	9	21	-2.074
9	22	-1.859	9	22	-1.818
12	8	-0.575	12	8	-0.576
12	10	-0.053	12	10	-0.053
13	7	-0.640	13	7	-0.642
13	8	-1.436	13	8	-1.438
13	9	-0.520	13	9	-0.521
13	11	-7.793	13	11	-9.499
13	12	-2.522	13	12	-3.067
14	8	-0.652	14	8	-0.642
14	12	-0.920	14	12	-1.117

TABLE 2.4

(Continued)

Water Head Observations, Fairfield - New Baltimore Aquifer System

Pumping Period = 5 (1956)

OBSERVATION POINT		DRAWDOWN (ft)
6	19	0
6	20	-1.427
6	21	-2.437
6	22	-2.274
7	19	-1.496
7	20	-3.306
7	21	-5.105
7	22	-3.609
8	19	-2.623
8	20	-5.271
8	21	-6.351
8	22	-4.441
9	19	-2.643
9	20	-3.992
9	21	-4.664
9	22	-4.434
12	8	-0.576
12	10	-0.053
13	7	-0.642
13	8	-1.438
13	9	-0.521
13	11	-8.987
13	12	-2.916
14	8	-0.654
14	12	-1.069

TABLE 2.4

(Continued)

Water Head Observations, Fairfield - New Baltimore Aquifer System

Pumping Period = 1 (1952)			Pumping Period = 2 (1953)		
OBSERVATION POINT		DRAWDOWN (ft)	OBSERVATION POINT		DRAWDOWN (ft)
6	19	-0.000000	6	19	-0.000000
6	20	-0.527064	6	20	-0.625530
6	21	-0.622973	6	21	-1.064879
6	22	-0.542445	6	22	-1.073707
7	19	-0.749563	7	19	-0.652465
7	20	-1.484903	7	20	-1.408121
7	21	-1.198743	7	21	-2.105527
7	22	-0.887911	7	22	-1.777399
8	19	-1.513599	8	19	-1.162237
8	29	-3.493261	8	20	-2.212571
8	21	-1.856055	8	21	-4.162179
8	22	-1.225276	8	22	-2.590137
9	19	-1.296270	9	19	-1.279244
9	20	-1.939404	9	20	-1.998344
9	21	-1.624880	9	21	-2.708470
9	22	-1.342505	9	22	-2.567733
12	8	-0.432524	12	8	-0.478767
12	10	-0.037855	12	10	-0.045582
13	7	-0.470057	13	7	-0.521750
13	8	-1.096278	13	8	-1.156436
13	9	-0.399910	13	9	-0.436903
13	11	-6.311068	13	11	-7.754892
13	12	-1.857041	13	12	-2.323575
14	8	-0.460457	14	8	-0.508858
14	12	-0.558667	14	12	-0.715042

TABLE 2.5

Water Head Predicted by the Model

Pumping Period = 3 (1954)			Pumping Period = 4 (1955)		
OBSERVATION POINT		DRAWDOWN (ft)	OBSERVATION POINT		DRAWDOWN (ft)
6	19	-0.000000	6	19	-0.000000
6	20	-0.873906	6	20	-0.672018
6	21	-1.719651	6	21	-0.887442
6	22	-1.462406	6	22	-0.853809
7	19	-0.667405	7	19	-0.880463
7	20	-1.746948	7	20	-1.754764
7	21	-3.820078	7	21	-1.584498
7	22	-2.209255	7	22	-1.299962
8	19	-0.892576	8	19	-1.733869
8	20	-1.597330	8	20	-3.834422
8	21	-2.322441	8	21	-2.288506
8	22	-2.045735	8	22	-1.694336
9	19	-0.919759	9	19	-1.566431
9	20	-1.386306	9	20	-2.311820
9	21	-1.766790	9	21	-2.071468
9	22	-1.867792	9	22	-1.831957
12	8	-0.484115	12	8	-0.484790
12	10	-0.046575	12	10	-0.046697
13	7	-0.528096	13	7	-0.528934
13	8	-1.162553	13	8	-1.163317
13	9	-0.440922	13	9	-0.441417
13	11	-7.471905	13	11	-9.109899
13	12	-2.248695	13	12	-2.735843
14	8	-0.514659	14	8	-0.515416
14	12	-0.695203	14	12	-0.846239

TABLE 2.5

(Continued)

Water Head Predicted by the Model

Pumping Period = 5 (1956)

OBSERVATION POINT		DRAWDOWN (ft)
6	19	-0.000000
6	20	-1.472941
6	21	-2.495643
6	22	-2.299668
7	19	-1.509039
7	20	-3.368410
7	21	-5.174324
7	22	-3.629240
8	19	-2.622289
8	20	-5.312718
8	21	-6.546117
8	22	-4.437453
9	19	-2.628840
9	20	-3.977404
9	21	-4.633173
9	22	-4.422836
12	8	-0.484879
12	10	-0.046712
13	7	-0.529048
13	8	-1.163418
13	9	-0.441482
13	11	-8.616365
13	12	-2.598289
14	8	-0.515519
14	12	-0.806728

TABLE 2.5

(Continued)

Water Head Predicted by the Model

The identification algorilthum was started using the second order polynomical approximation of the transmissivily function, $T(x,y)$.

$$T(x,y) = b_1 x^2 + b_2 y^2 + b_3 x + b_4 y + b_5$$

The initial estimates of the parameters to be identified - b_1, b_2, b_3, b_4, and b_5 , were as follows:

$$b_1 = .60 \times 10^{-13}$$
$$b_2 = .30 \times 10^{-13}$$
$$b_3 = .60 \times 10^{-8}$$
$$b_4 = .10 \times 10^{-9}$$
$$b_5 = .01$$

The identification scheme was very efficient computationally. The least square error, ϕ ($\underline{b}$), is driven to a minimum value of 0.8 $ft.^2$ in only five iterations. As expected, the initial estimate of parameter values plays a dominant role in computation time. However, the least squares error ϕ ($\underline{b}$)converages quadratically to a minimum even with "bad" initial values (i.e. corresponding to a large initial least square error). The model-predicted drawdowns for the years 1952 to 1956 are shown in Table 2.5, demonstrating a very satisfactory representation.

The results of each successive approximation are tabulated in Table 2.6. Note that the convergence is quadratic, after only five iterations the magnitude of the sum of the squared deviations between observed and theoretical values is approximately 0.8 $ft.^2$ Figure 2.3 shows the relationship of the error function ϕ ($\underline{b}$) to the number of iterations.

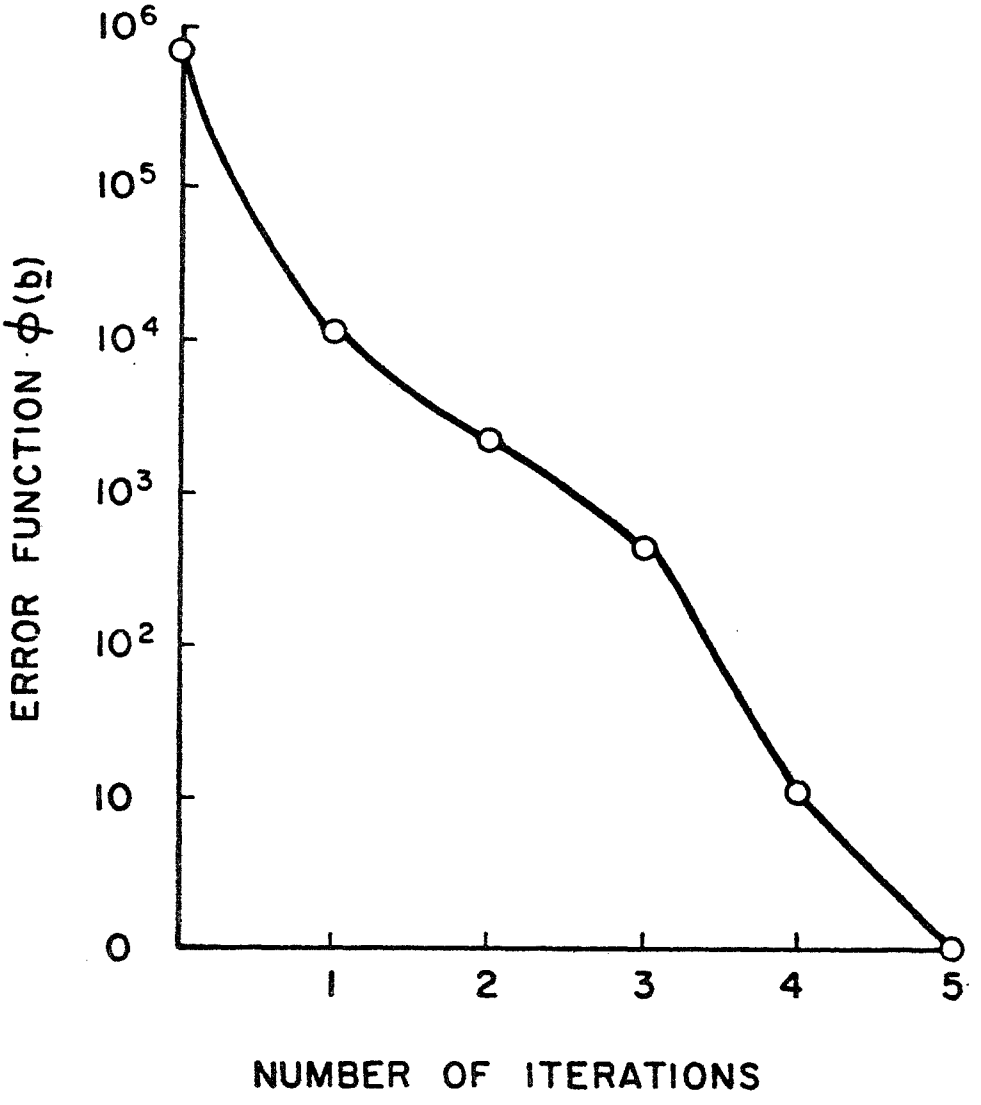

FIGURE 2.3

Plot of Error Norm vs. Number of Iterations

Parameters	Initial Estimate	ITERATION NUMBER 1	2	3	4	5
b_1	$.60\times10^{-13}$	.8085x15''	$.8679\times10^{-11}$	$.131511\times10^{-12}$	$.84449\times10^{-12}$	$-.25293\times10^{-10}$
b_2	$.30\times10^{-13}$	$.52074\times10^{-11}$	$-.41101\times10^{-11}$	$-.71359\times10^{-12}$	$.11201\times10^{-12}$	$.88937\times10^{-10}$
b_3	$.60\times10^{-8}$	$.26177\times10^{-6}$	$.9600\times10^{-6}$	$.81137\times10^{-5}$	$.94568\times10^{-5}$	$.90333\times10^{-5}$
b_4	$.10\times10^{-9}$	$.20536\times10^{-6}$	$.5072\times10^{-6}$	$.13942\times10^{-6}$	$.24862\times10^{-5}$	$.40493\times10^{-5}$
b_5	.01	.018616	.045449	.099135	.16848	.1573708
Error Function $\phi(b)$ $(ft.^2)$	$.9524\times10^{6}$	$.1455\times10^{5}$	$.3935\times10^{4}$	$.61921\times10^{3}$	$.1155\times10^{2}$	.7996

TABLE 2.6

Results of the Identification of the Fairfield-New Baltimore Aquifer System

2.2.2 Model Validation

Once the Parameters defining the transmissivity function have been estimated, the appropriate next test of the calibrated equipment model is its prediction of the aquifer response to any demand placed on it. The model validation was carried out by simulating the aquifer described by equation (1.2) for the 10 years from 1952 to 1962. These predicted results were then compared with the observed real system response during 1962 [Spieker, 1968]. The results of model validation are displayed in Table 2.7. The drawdown predicted by the model at well locations are compared with observed data as a measure of accuracy of the model. A comparison of this result with that obtained in Phase I showed a significant improvement in predictive capability. It seems reasonable to attribute this improvement partly to the modified mathematical structure of the transmissivity function, and partly to the improved convergence of the identification algorithum.

2.3 Sensitivity Analysis:

2.3.1 Introduction

In addition to the optimum solution of the model, it is desirable to generate additional information concerning the response of the solution to changes in the systems parameters. This is referred to as sensitivity analysis, or as post optimality analysis. This type of analysis is particualarly important whenever the parameters of the system cannot be precisely estimated or measured.

Well Coordinates	Well Name	Observed Head	Predicted Head Phase II work	Error %	Predicted Head Phase I work	Error %
(13,8)	A-2	6.0	4.15	30.83	4.0	33
(13,11)	S1-S2	15.0	12.0	20	12.0	20
(8,20)	F-16	6.5	6.14	5.6	7.7	18.4
(8,21)	F-11	6.5	7.40	13.9	8.7	30
(7,21)	F-10	6.5	6.05	6.93	7.5	15.3
(8,19)	BU-7	4.0	3.15	21.2	3.13	21.7

TABLE 2.7

Results of the Fairfield - New Baltimore Aquifer Model Validation

2.3.2 Effect of Changes in Storativity on Model Calibration

A sensitivity analysis was performed to determine the effect of changes in storativity on the parameter values identified and on the model calibration. This is particularly important since the value of storage coefficient varies from 0.1 to 0.2 over the aquifer modeled [Spieker, 1968] and the digital model used to simulate the aquifer system provides only for a homogeneous storage coeificient (an average value of 0.145 is used both for simulation and identification of the model). The values of the optimally estimated parameters and behavior of this optimal solution are studied in the neighborhood of a storage coefficient value of 0.145. This provides information on the behavior of the error function ϕ ($\underline{b}$), and consequently model caliberation in this neighborhood. The model is calibrated for two storativity values: 0.12 and 0.2. The results of this analysis is displayed in Table 2.8.

It is noted that with different values of the storage coefficients the algorithim performs quite efficiently to reduce the error norm $\phi(\underline{b})$ to a minimum value. The optimal parameters $\underline{b}$, are found to be different in each case (see Table 2.8) As a secondary result of this sensitivity analysis, it is very encouraging to note that the identification algorithim developed provides for compensations and/or adjustments in the transmissivity parameter values in the event of error or other mis-approximation in the other model parameters, e.g. storage coefficient. The model response seems to adequately represent the real system.

2.3.3 Effect of Errors in Observed Drawdown on Model Calibration

A sensitivity analysis was also performed to evaluate the effect on optimal identification and model calibration of errors in the observed output, i.e., the recorded drawdowns. The identification problem was rerun with error artifically introduced in drawdown at well locations S-2 (See Figure 1.1). This yields the effect of this error on the optimal parameter values and on the error norm ϕ ($\underline{b}$), is presented in Table 2.9.

We note that in this case also, the algorithum provides for adjustment of optimal transmissivity parameter values. Thus the output error function ϕ ($\underline{b}$) is driven toward a minimum, and the model is calibrated. This once again demonstrates the capability of the identification algorithum to compensate for inadvertant errors introduced elsewhere in the model.

Optimal Parameters	Storage Coefficient (S)		
	S = 0.12	S = 0.145	S = 0.20
b_1	$.45766 \times 10^{-11}$	$-.25293 \times 10^{-10}$	$-.50372 \times 10^{-11}$
b_2	$-.9061265 \times 10^{-10}$	$.88937 \times 10^{-10}$	$-.1421769 \times 10^{-9}$
b_3	$.4090793 \times 10^{-5}$	$.90333 \times 10^{-5}$	$.455979 \times 10^{-5}$
b_4	$.631229 \times 10^{-5}$	$.40493 \times 10^{-5}$	$.5323699 \times 10^{-5}$
b_5	.3492658	.1573708	.3617476
Minimized Error Function ϕ ($\underline{b}$)	1.3091	.7996	1.295

TABLE 2.8

Results of Sensitivity Analysis:

The Effect of Changes in Storativity on Model Calibration

Time Period	Observation Wells, S-2		Optimal Parameters
	Real Observation	Error Introduced (%)	
1	-6.575	-10%	$b_1 = -.19310 \times 10^{-10}$
2	-8.086	- 7%	$b_2 = -.91957 \times 10^{-11}$
3	-7.793	+15%	$b_3 = .88197 \times 10^{-5}$
4	-9.499	+ 8%	$b_4 = .71241 \times 10^{-5}$
5	-8.987	+ 5%	$b_5 = .1871252$
Minimized Error Function ϕ ($\underline{b}$)	.7996	1.208	

TABLE 2.9

Results of Sensitivity Analysis:

The Effect of Errors in Observed Drawdown on Model Calibration

Chapter 3

A RESPONSIVE MODEL FOR A COMPLEX AQUIFERS-STREAMS SYSTEM

Application of the Superposition Approach

3.1 Introduction

In Phase I the application of the superposition approach as a modeling procedure for a multicell aquifer groundwater system was introduced. The multicell approach to modeling of groundwater makes use of a set of balance equations, each representing a mass balance applied to a particular cell. This provides approximate inflows and outflows for each cell. To obtain an accurate estimate of drawdown at a certain point of interest, one can isolate the aquifer cell to which this point belongs. This cell may then be modeled via a more detailed mathematical model, taking into account the particular (time variant) boundary conditions relating to its adjacent cells. In addition to the multicell-particular cell modeling procedure, the algebraic technological functions (A.T.F.) concept was introduced. Extension of these functions to include a complex groundwater system consisting of multicell aquifers allows the system to be coupled with any desired management model. The reason for extension of and emphasis on this work is that it makes available an extremely powerful tool for optimal management of complex water systems.

The case study used in Phase I to illustrate the usefulness of the proposed modeling technique was the Fairfield-New Baltimore area in the lower Great Miami River Valley. In Phase II of this study a more sophisticated mathematical analysis of the proposed superposition approach, via the A.T.F., is obtained. In order to consider interactions between streams and aquifers, the algebraic function relating the infiltration rate (from a stream into an aquifer) to pumpage from wells is introduced. Extension of this function to a complex system (again applying the superposition technique) provides the coupling with any desired ground and surface water management model.

To complete the development of the responsive model, the present work considers the case where a possible change in the boundary conditions between the stream and aquifer is induced during the planning period.

3.2 The Algebraic Technological Functions.

If the vertical flow of groundwater is insignificant, such as for wells fully penetrating an artesian aquifer or a phreatic aquifer in which the drawdown always remains small with respect to the saturated thickness, Maddock [1972] shows that it is possible to construct algebraic technological functions that relate pumpage to drawdown at wells. The functions exist for aquifers with irregularly shaped boundaries and nonhomogeneous flow parameters, (see Appendix I).

The drawdown at a point $\hat{x} = (x,y)$, $d(\hat{x},t)$ is given by the partial differential equation:

$$\frac{\partial}{\partial \hat{x}}\left[T(\hat{x},t)\frac{\partial}{\partial x}[d(\hat{x},t)]\right] = S(\hat{x})\frac{\partial}{\partial t}[d(\hat{x},t)] - \sum_{j=1}^{M} q(\hat{x}_j,t)\,\delta(\hat{x}-\hat{x}_j) \qquad (3.1)$$

where $T(\hat{x},t)$ and $S(\hat{x})$ are transmissivity and storage coefficients respectively, $\delta(\hat{x}-\hat{x}_j)$ is a Dirac Delta Function, $\hat{x}_j$ indicates the position of the j^{th} well and $q(\hat{x}_j,t)$ is the instantaneous discharge at the j^{th} well. M is the total number of wells.

For a given set of initial and boundary conditions, the drawdown $d(\hat{x},t)$, at any point $\hat{x}$ within defined boundaries at any time t is

$$d(\hat{x},t) = \sum_{j=1}^{M} \int_{o}^{t} G(\hat{x},\hat{x}_j,t-\tau)\, q(\hat{x}_j,\tau)\,d\tau \qquad (3.2)$$

where $G(\hat{x},\hat{x}_j,t-\tau)$ is the Green's function for equation (3.1) satisfying the particular initial and boundary conditions.

If t is divided into n time units such that the planning horizon T comprises exactly N time units, equation (3.2) may be represented by an algebraic technological function (A.T.F.), $d(k,n)$

$$d(k,n) = \sum_{j=1}^{M}\sum_{i=1}^{n} \beta(k,j,n-i+1)\, q(j,i) \qquad (3.3)$$

This equation determines the drawdown at the k^{th} well at the end of the n^{th} time period due to the pumping of M wells. $q(j,i)$ is the quantity of water pumped by the j^{th} well over the i^{th} time period, and $\beta(k,j,n-i+1)$ is the response matrix for the n^{th} time period relating the drawdown at well k to unit pumping at well j during the time period i. The parameters beta are related to the Green's function, having the advantage of the algebraic formulation characterizing the technological function. However the beta parameters are not given explicitly by their derivation, and Maddock [1969] proposes to achieve the beta calculation using a digital simulation model of the aquifer.

There are two possible procedures for producing the betas. One, where the simulation model already includes the boundary values explicitly, may provide the calculation of the real steady state head distribution throughout the system. Then the procedure for the beta calculation is as follows:

(i) Determine the initial state before pumpage began. Running the aquifer digital model over a long time with no pumpage imposed provides a steady state head distribution for the system independent of the (randomly chosen) initial values.

(ii) During the first period after reaching the steady state, one unit of pumpage is imposed on the model at one particular well. No pumpage is imposed after the first period. The values of drawdown determined by the model correspond to the beta values relating drawdown at wells due to pumpage at the particular well, at the end of the first period and at the proceeding periods.

(iii) The procedure described in (ii), when applied one by one to each of the wells in the system, provides the determination of the desired matrix of beta values.

The second procedure for determination of beta values is in a sense more general, making use of the linear characteristics of the model.

Let $H(\hat{x})$ be the steady state head distribution throughout the system before pumpage began. Let $h(\hat{x},t)$ be the head distribution above the same reference used to measure $H(\hat{x})$. Then the drawdown $d(\hat{x},t)$ is:

$$d(\hat{x},t) = H(\hat{x}) - h(\hat{x},t) \tag{3.4}$$

Applying (3.4) to (3.1) yields:

$$\frac{\partial}{\partial \hat{x}}\left[T(\hat{x},t)\frac{\partial}{\partial \hat{x}}\left[h(\hat{x},t)\right]\right] = S(\hat{x})\frac{\partial}{\partial t}\left[h(\hat{x},t)\right] + \sum_{j=1}^{M} q(\hat{x}_j,t)\delta(\hat{x}-\hat{x}_j) \qquad (3.5)$$

Let the no-flow boundary conditions be $\frac{\partial}{\partial n} d(\lambda,t) = 0$, where λ is a parameter indicating that $\frac{\partial d}{\partial n}$ is evaluated on the boundary (which is irregular in shape). The n is the normal direction and the $\frac{\partial d}{\partial n}$ is the gradient of the drawdown for the normal to the boundary, which vanishes under no-flow conditions.

Let the constant head boundary conditions be $d(\mu,t) = C$(Constant) where μ is a parameter indicating that the drawdown d is evaluated along the constant head boundary. Applying (3.4) to the set of boundary conditions:

$$\frac{\partial h}{\partial n}(\lambda,t) = 0 \qquad h(\mu,t) = H(\mu) = C$$

If we choose the reference for measuring $h(\hat{x},t)$ such that $h(\mu,t) = 0$. Then $d(\mu,t) = h(\mu,t) = H(\mu) = 0$.

The steady state solution corresponds to the solution of

$$\left.\begin{array}{l} \frac{\partial}{\partial \hat{x}}\left[T(\hat{x},t)\frac{\partial}{\partial \hat{x}}\left[h(\hat{x},t)\right]\right] = 0 \\ \text{and B.C.} \\ h(\mu,t) = 0 \qquad \frac{\partial h}{\partial n}(\lambda,t) = 0 \end{array}\right\} \quad h(\hat{x}, \text{ Steady State}) = h(\mu,t) = 0 \qquad (3.6)$$

The conclusion is that if the boundary conditions are independent of drawdowns, the drawdown due to pumpage is independent of the assigned boundary values, i.e. we can choose $d(\mu,t) = 0$. The steady state drawdown distribution will be:

$$d(\hat{x}, \text{Steady State}) = 0$$

Assigning these values to the boundary conditions in the simulation model the drawdown is equal to the head distribution, and the procedure for calculating beta's is essentially similar to steps (ii) and (iii) in the first procedure.

3.3 Extension of the Algebraic Technological Functions (A.T.F.) Toward a Complex Aquifer System - The Superposition Approach.

In this study we are primarily interested in the coupling of a groundwater system with a desired management model. Using the proposed technique, the A.T.F. approach can be extended to handle a more complex system. The basis used is that the drawdown at a point in an aquifer due to input at some other points within the groundwater system may be approximated by a linear combination of responses predicted by the multicell model and those predicted by the more detailed model of the unit aquifer to which the point belongs.

For the aquifer system where the drawdown $d(\hat{x},t)$ is given by (3.5) and boundary conditions, as defined in Section 3.2, the solution is approximated by: (See Appendix A)

$$d(\hat{x},t) = \sum_{j=1}^{M} \int_{\sigma}^{t} \alpha(\hat{x},\hat{x}_j,t-\tau)\, q(\hat{x}_j,\tau)\, d\tau \qquad (3.7)$$

Divide the aquifer system into R different cells. The drawdown $d(\hat{x}_r,t)$ at a point $\hat{x}_r$ located at the r^{th} cell is given by the partial differential equation (PDE):

$$\frac{\partial}{\partial \hat{x}}\left[T(\hat{x}_r,t)\frac{\partial}{\partial \hat{x}}\left[d(\hat{x}_r,t)\right]\right] = S(\hat{x}_r)\frac{\partial}{\partial t}\left[d(\hat{x}_r,t)\right] + \sum_{j=1}^{M} q(\hat{x},t)\,\delta(\hat{x}_r-\hat{x}_j) \qquad (3.8)$$

with boundary conditions

$$\frac{\partial}{\partial n} d(\lambda,t) = 0$$

and $d(\mu,t) = 0$

and the initial conditions $d(\hat{x},o) = 0$.

The solution for (3.8) is assumed to be

$$d(\hat{x}_r,t) = \sum_{j=1}^{M} \int_0^t \alpha(\hat{x}_r,\hat{x}_j,t-\tau)\ q(\hat{x}_j,\tau)d\tau$$

$$= \sum_{j \text{ in } r} \int_0^t \alpha(\hat{x}_r,\hat{x}_j,t-\tau)\ q(\hat{x}_j,\tau)d\tau$$

$$+ \sum_{j \text{ not in } r} \int_0^t \alpha(\hat{x}_r,\hat{x}_j,t-\tau)q(\hat{x}_j,\tau)d\tau \qquad (3.9)$$

The first term in the right of (3.9) stands for the drawdown at $\hat{x}_r$ due to pumping inside cell r. The second term stands for the drawdown at x_r due to pumping from other cells. Consider the term

$$d_1(\hat{x}_r,t) = \sum_{j\varepsilon r} \int_0^t \alpha(\hat{x}_r,\hat{x}_j,t-\tau)q(\hat{x}_j,\tau)d\tau \qquad (3.10)$$

Equation (3.10) is also the solution of the P.D.E. applied to the isolated cell system:

$$\frac{\partial}{\partial x_r}\left[T(\hat{x}_r,t)\frac{\partial}{\partial x_r}\left[d(\hat{x}_r,t)\right]\right] = S(\hat{x}_r)\frac{\partial}{\partial t}\left[d(\hat{x}_r,t)\right]$$

$$+ \sum_{j=1}^{m_r} q(\hat{x}_{rj},t)\delta(\hat{x}_r-\hat{x}_{rj}) + q(\xi_r,t)\delta(\hat{x}_r-\xi_r) \qquad (3.11)$$

$q(\xi_r,t)$ is the time varying flow conditions on the boundaries denoted by ξ_r between the r^{th} cell and neighboring cells, and are regarded as flow forcing functions on the system along ξ_r.

The P.D.E. (3.11) is subject to the natural constant head and no-flow boundary conditions associated with the r^{th} cell area.

To solve for $q(\xi_r,t)$, the approximation of the finite difference multicell model may be used (Phase I):

$$\sum_{\ell} \left\{ R_{\ell,r}\left[h(\ell,i) - h(r,i)\right] + U_{\ell,r}\left[(h(\ell,i))^2 - (h(r,i))^2\right] \right\}$$

$$= V_r\left[h(r,i+1) - h(r,i)\right] + Q(r,i) \qquad (3.12)$$

where

$$R_{\ell,r} \triangleq \frac{W_{\ell,r}\, C_{\ell,r}}{L_{\ell,r}}, \quad U_{\ell,r} \triangleq \frac{W_{\ell,r}\, k_{\ell,r}}{2L_{\ell,r}}, \quad V_r \triangleq \frac{A_r\, S_r}{\Delta t}$$

$$C_{\ell,r} \triangleq K_{\ell,r}(E_{\ell,r} - F_{\ell,r})$$

$h(\ell,i)$ = water table elevation at the ℓ^{th} cell during the i^{th} time step.

$Q(r,i)$ = net outflow from the r^{th} cell during the i^{th} time step.

$W_{\ell,r}$ = length of the perpendicular sector associated with the segment between cells ℓ and r.

$L_{\ell,r}$ = distance between the centers of nodes ℓ and r.

$K_{\ell,r}$ = hydraulic conductivity averaged between cells ℓ and r.

$E_{\ell,r}$ = effective aquifer depth averaged between cells ℓ and r.

$F_{\ell,r}$ = elevation at the top of the aquifer averaged between cells ℓ and r.

A_r = area of r^{th} cell.

S_r = storage coefficient averaged over the r^{th} cell.

On the boundary between the neighboring cells r and ℓ, the inflow to $r, q(\xi_r, i)$ at time period i is:

$$R_{\ell,r}\left[h(\ell,i) - h(r,i)\right] + U_{\ell,r}\left[(h(\ell,i))^2 - (h(r,i))^2\right] \tag{3.13}$$

Using the same arguments as in Appendix I, Equation (3.10) becomes

$$d_1(\hat{x}_r,t) \simeq d_r(k,n) = \sum_{j=1}^{m_r}\sum_{i=1}^{n} \beta_r(k,j,n-i+1)q_r(j,i) \tag{3.14}$$

Equation (3.14) expresses the drawdown at the k^{th} well located at the r^{th} cell at the end of the n^{th} period.

$\beta_r(k,j,n-i+1)$ is the drawdown at k at the end of the n^{th} period due to one unit input at j during the i^{th} period, with k and j located at the r^{th} unit aquifer.

The procedure for determining the betas is discussed in detail in Section 3.2. To produce the $\beta_r(k,j,n-i+1)$'s, the particular r^{th} cell model may be used. This model is isolated from the rest of the system by accounting for the time varying flow conditions along the boundaries ξ_r. These flow conditions are associated with the pumpage of one unit from the r^{th} cell during the i^{th} period, and are determined directly by the multicell model.

The second term in equation (3.9)

$$d_2(\hat{x}_r,t) = \sum_{j\neq r} \int_0^t \alpha(\hat{x}_r,\hat{x}_j,t-\tau)q(\hat{x}_r,\tau)d\tau \tag{3.15}$$

may be considered under the aggregation of the pumpage at the different cells not including the r^{th} cell. We herein assume, that pumping outside the r^{th} cell causes the same drawdown for all the wells inside this cell. Hence:

$$d_2(\hat{x}_r,t) \simeq D(r,n) = \sum_{\substack{\ell=1\\ \ell\neq n}}^{R}\sum_{i=1}^{n} \delta(r,\ell,n-i+1)q(\ell,i) \tag{3.16}$$

where $\gamma(r,\ell,n-i+1)$ is the average drawdown at cell r in the n^{th} period due to one unit pumpage in cell ℓ during the i^{th} period, and $q(\ell,i)$ is the total quantity of water pumped from cell ℓ during period i.

Substitute (3.14) and (3.16) into (3.9):

$$d(\hat{x}_r,t) \cong DT_r(k,n) = \sum_{j=1}^{m_r} \sum_{i=1}^{n} \beta_r(k,j,n-i+1)q_r(j,i)$$

$$+ \sum_{\substack{\ell=1 \\ \ell\neq r}}^{R} \sum_{i=1}^{n} \gamma(r,\ell,n-i+1)q(\ell,i) \qquad (3.17)$$

Equation (3.17) states the algebraic function for the drawdown at point k in an aquifer affected by pumpage in the close vicinity and by pumpage from wells located at other parts of the aquifer system.

The advantages associated with the development of the function (3.17) may now be realized. It is possible to use this function for the purpose of coupling the more complex groundwater system with many classes of management models regarding the management of water resources systems. The dimension of the matrices of A.T.F. functions (to be determined and stored for later use in the management model formulation) is substantially reduced, as is the amount of data necessary for identifying the aquifer system for the model. Its use will be illustrated in the development of management models included in this study.

3.4 A Case Study For the Applications of the Modified A.T.F.

In this section our purpose is to illustrate the advantages of the superposition of groundwater response via the A.T.F. functions developed in Section 3.3. In particular, the A.T.F. functions may be used for prediction of drawdowns due to pumping in a complex groundwater system. The basis used is that drawdown at a point in an aquifer due to input at some other point within the groundwater system may be approximated by a linear combination of responses predicted by some crude overall model (multicell), and those predicted by the more detailed model of the unit aquifer to which the point belongs (the particular cell). The procedure for achieving it is:

1. Use the multicell model to calculate A.T.F. functions between cells (gamma functions).

2. Use the particular cell detailed model to produce algebraic technological functions (A.T.F.) between points within the cell (beta functions). The same unit input is to be used for the process of producing the functions in both models. Beta functions within the cell are thus calculated under boundary conditions provided by the multicell model. These result from heads and inflows predicted from the multicell model due to a unit pumpage imposed on the particular cell.

3. The drawdown at a point inside the cell of interest is calculated from the given inputs and functions in the manner described in previous sections.

To illustrate the usefulness of the proposed modeling technique, the Fairfield-New Baltimore aquifer in the lower Great Miami River Valley was modeled. A computer program was written to simulate the aquifer. (Appendix B) The system was divided into cells with differing characteristics (see Fig. 3.1). The data utilized include pumpage, water elevations, and cell boundary conditions, and were taken from Spieker, (Spieker, 1968). A more detailed description of the case study area is given in Phase I of this study.

Cell 4 (see Figure 3.2) was selected to be simulated by means of a particular model. Maddock's program (Maddock, 1969) was used for this purpose.

To show the possible applications of this procedure to the case of study, beta functions were calculated for three wells located in the cell. The boundary conditions were taken from the results of the multicell model, where unit pumpage was imposed on Cell 4 to calculate the related A.T.F.

(gamma) functions. Table 3.1 summarizes the A.T.F. functions determined for the three wells. Table 3.2 lists the pumpage data in the wells during five years (1956 - 1962).

$\beta(k,j,i)$ Values $\left[Ft/Ft^3/Day\right]$ *1000
Wells in Cell 4

Year I	(F-10,J,I)			(F-11,J,I)			(F-16,J,I)		
	J			J			J		
	F-10	F-11	F-16	F-10	F-11	F-16	F-10	F-11	F-16
1	10.00	4.77	2.99	4.82	11.51	4.74	3.05	4.77	9.82
2	0.98	1.04	0.74	1.01	1.32	0.94	0.73	0.95	0.83
3	0.24	0.27	0.19	0.26	0.31	0.23	0.18	0.22	0.17
4	0.07	0.09	0.06	0.08	0.08	0.06	0.05	0.06	0.04
5	0.02	0.02	0.02	0.02	0.02	0.02	0.01	0.01	0.01

TABLE 3.1

Pumpage From Wells in Cell 4
(Years 1956-1962)
$\left[Ft^3/Day\right]*10^{-3}$

Year I	F-10	F-11	F-16
1	325	407	325
2	330	412	330
3	321	402	321
4	307	385	307
5	307	385	307

TABLE 3.2

Using the A.T.F. functions, drawdowns at these wells were computed. Notice, that Cell 4 is separated from Cell 5 by the river (Cell 10). In the studied area, however, wells are located in Cells 4 and 5 only. Thus to determine drawdowns at wells located in Cell 4, only natural flow into cell 2 has to be considered. The beta functions provide the desired solution directly (note that the beta functions include the natural influence of the outside system through the boundaries). Table 3.3 summarizes drawdowns at wells computed via the A.T.F. approach, compared with corresponding values via other techniques.

Drawdown (In Feet)
For Wells in Cell 4
(Year 1962)

Well	Real	Analog	Digit	2 Stage	A.T.F.
F-10	6.50	9.0	7.6	6.8	7.0
F-11	6.50	9.0	9.1	8.5	8.8
F-16	6.50	9.0	7.8	7.2	6.9

TABLE 3.3

To show possible applications of the superposition approach when pumpage does occur in some neighboring cell and thus may influence drawdown in a cell of interest, a synthetic pumpage imposed on Cell 2 while no change in input was assumed at Cell 4. Two techniques were considered:

I. 2-Stage Simulation
II. Superposition of the Response.

Again, the years 1956 - 1962 were considered. A constant annual pumpage of 500,000 Ft^3/Day was imposed on Cell 2, in addition to the given pumpage at wells in Cell 4.

The 2-Stage simulation procedure required "running" the multicell model with the modified input, and then "running" the cell 4 model, taking into account the new resulting boundary conditions.

To determine the system response to the new input imposed on it, gamma functions provide the influence of the pumpage in Cell 2 on drawdown in Cell 4. This is equivalent to computing the influence of the boundary conditions on drawdowns inside the cell. Adding these values to the particular drawdown as calculated before, the total expected response at each well is easily calculated. Table 3.4 summarizes the results.

Drawdown in Feet
(At the end of 5 years)

Pumpage is Added in Cell 2

Well	Simulation	Superposition
F-10	7.6	8.2
F-11	9.2	9.4
F-16	7.7	8.1

The possible error due to the linearization compared with the simulation is within 10%.

TABLE 3.4

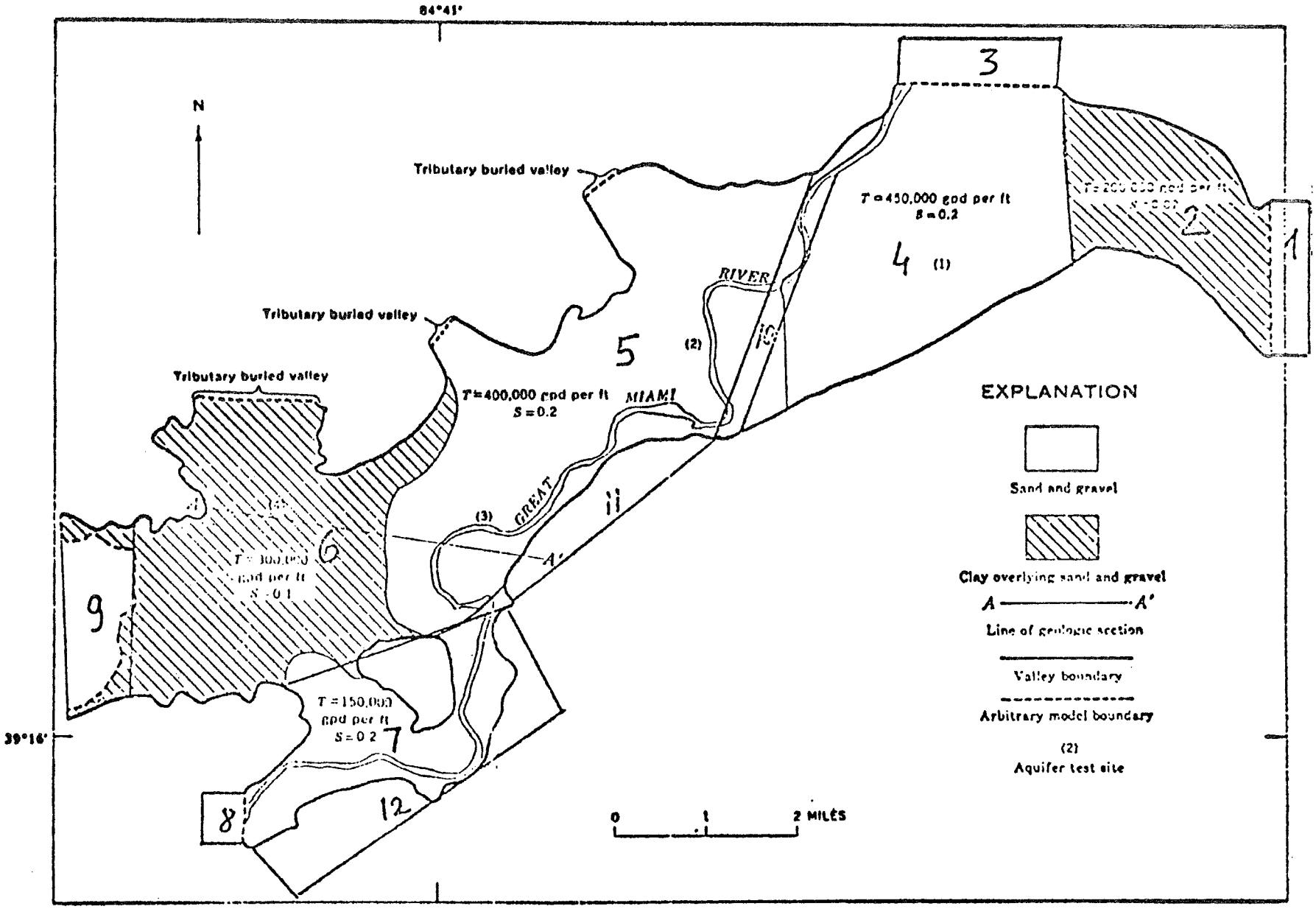

FIGURE 3.1

Analog Study of Increased Pumping Effects, Fairfield-New Baltimore Area

Generalized geology and coefficients of transmissibility (T) and storage (S) of the Fairfield-New Baltimore area. Cell's assignment--Cells 10,11, and 12 represent the river.

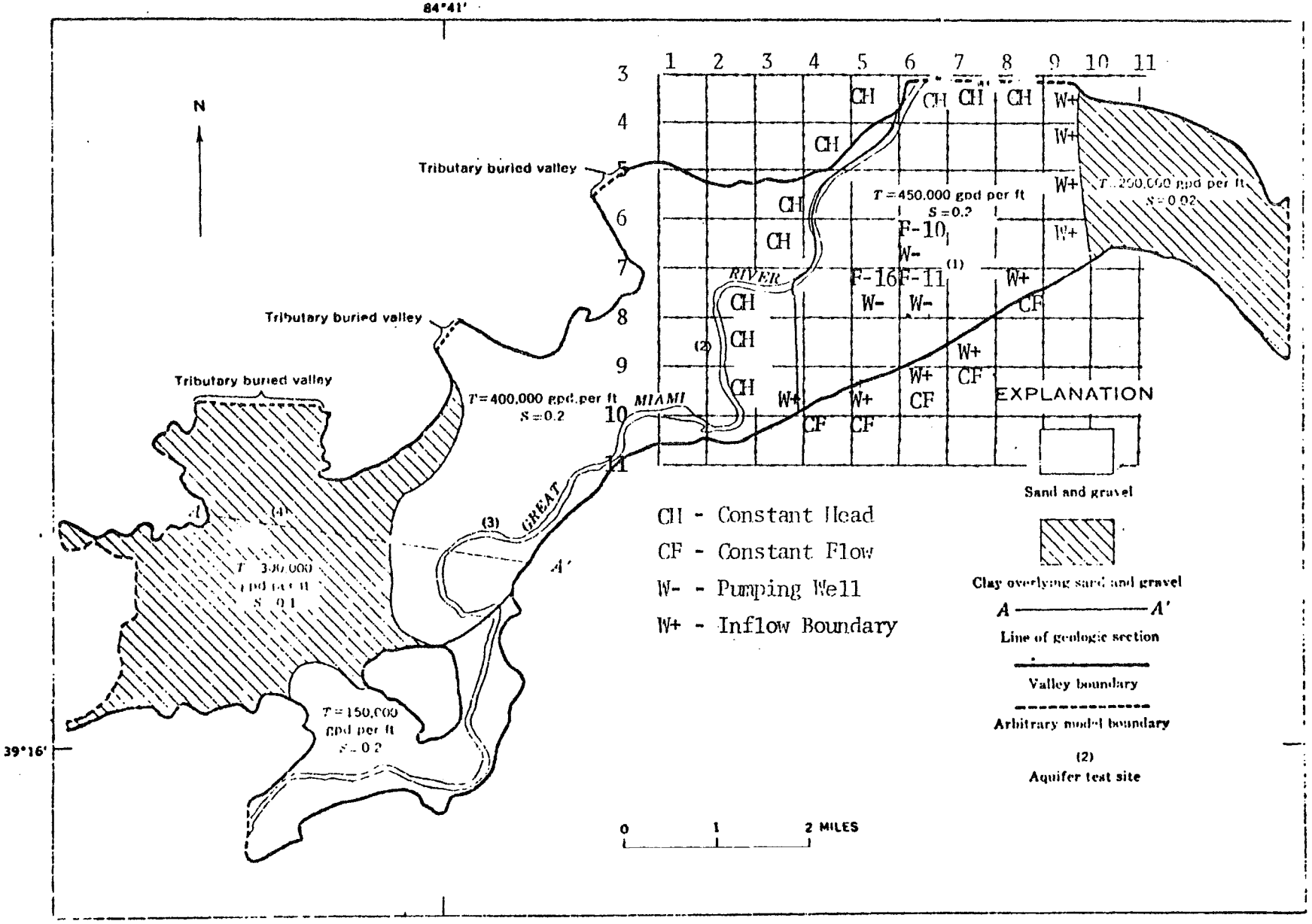

FIGURE 3.2

Generalized geology and coefficients of transmissibility (T) and storage (S) of the Fairfield-New Baltimore area. Cell 4 discretization for the detailed modeling.

3.5 The Stream-Aquifer Flow Functions

3.5.1 General Discussion.

The basic objective of streamflow simulation models is to predict with reasonable accuracy the response of the watershed to an input of precipitation (Ricca, V.T., 1972). Our study, however, is devoted to the more specific question of predicting the joint response of surface water and groundwater systems to artificially controlled forcing functions imposed on the aquifers. In particular, when the water resource system comprises a network of streams traversing a system of multiple aquifer cells, we may be interested in the response of the surface into ground flow to pumpage from the aquifers. In preceding sections we have widened our discussion to include the inverse problem, relating the groundwater response to changes in the stream-aquifer relations through the physical boundaries.

It is understood that when considering a streamflow system, the stream network is only one component within the hydrologic model (Haimes, et al., 1973); the interrelations between streams and aquifers are only a subset of the set of active relations existing in the complex. Based on these arguments, the following assumptions are made:

(i) There is "sufficient" knowledge concerning the streams in the network to produce models aiming at the determination of the time-varying flow balance at each stream.

(ii) The necessary input data for the flow balance model is available, and the only other information necessary to completely determine the flow balance are the infiltration rates into and from the ground water effecting this balance.

3.5.2 Pumping From Groundwater to Stream-Aquifer Interflow Functions.

In the case where interflows between streams and aquifers play a substantial role in the water flow balance of these streams, it is desired to mathematically describe the dynamic responsive nature of both systems to inputs imposed on either.

Considering the case of pumpage from an aquifer hydraulically connected with a stream, Maddock, (1974) developed a set of technological functions that relate the drawdowns and interactions between the stream and aquifer produced by wells. The following conditions must hold:

(i) Vertical flow of groundwater is insignificant (wells fully penetrating).

(ii) The groundwater system consists of an artesian aquifer or a water table aquifer in which the drawdown always remains small with respect to the saturated thickness.

Under these conditions, if a stream traverses the aquifer, and may be considered as a constant head boundary for the groundwater model, the drawdown at a point $\hat{x} = (x,y)$, $d(\hat{x},t)$, is given in Section 3.2 (Equations (3.1) and (3.2) with boundary and initial conditions as in (3.6)).

Multiplying Equation (3.2) by the storage coefficient and integrating over the surface area of the aquifer gives the quantity of water $F_s(t)$ removed from aquifer storage over time t:

$$F_s(t) = \sum_{j=1}^{M} \int_0^t \left[\iint_{\substack{\text{over} \\ \text{all } \hat{x}}} S(\hat{x}) G(\hat{x},\hat{x}_j,t-\tau) d\hat{x} \right] q(\hat{x}_j,\tau) d\tau \qquad (3.18)$$

The quantity of water removed from storage and the river over t is:

$$F_{sr}(t) = \sum_{j=1}^{M} \int_0^t q(\hat{x}_j,\tau) d\tau \qquad (3.19)$$

Hence the quantity of water removed from the river by interaction over t is:

$$F(t) = F_{sr}(t) - F_s(t) = \sum_{j=1}^{M} \int_0^t \left[1 - \iint_{\hat{x}} S(\hat{x}) G(\hat{x},\hat{x}_j,t-\tau) d\hat{x} \right] q(\hat{x}_j,\tau) d\tau \qquad (3.20)$$

Under the assumption that the quantity of water pumped may vary from time period to time period, the quantity of water induced to flow from the river into the aquifer is:

$$f(n) = \begin{cases} F(1) & n = 1 \\ F(n) - F(n-1) & n > 1 \end{cases} \qquad (3.21)$$

where

$$F(n) = \sum_{j=1}^{M} \sum_{i=1}^{n} \rho(j,n-i+1)q(j,i) \qquad (3.22)$$

$\rho(j,n-i+1)$ is the fraction of water pumped from well j at time i supplied by the river, during the period from i to n. More details are in Maddock, (1974).

$F(n)$ is the distribution function of the stream-aquifer interaction response to pumping in wells. It is more useful for our purpose to define the density function $\phi(j,n-i+1)$ to represent the quantity of water induced from the stream into the aquifer during the n^{th} period due to one unit pumpage at the j^{th} well during the i^{th} period.

$f(n)$ is the quantity of water induced from the river into the aquifer at the n^{th} time period:

$$f(n) = \sum_{j=1}^{M} \sum_{i=1}^{n} \phi(j,n-i+1)q(j,i) \qquad (3.23)$$

If the stream traverses the aquifer, but the groundwater table is below the level where it may effect the infiltration, the quantity of water induced from the stream into the aquifer is considered mainly as a function of the water level in the stream above its bed (neglecting other effects like water temperature).

3.5.3 Linear Superposition of Stream-Aquifer Interflow Functions.

Consider a network of streams traversing a complex aquifer system, regarding them as constant head boundaries for the aquifers. The groundwater system is divided into several cells according to hydrological, geological, political or other considerations. The stream network may also be divided into many reaches independently of the cell boundaries. We begin by stating the logical basis for the following discussion. Quantities of water flowing between stream units and groundwater units under steady state conditions may be determined and referred to as initial flow conditions. Pumpage at a well disturbs this basis state, and induces a flow from each reach of every stream into the aquifers traversed by these streams. The induced flow from several streams into one cell, as well as from a single stream into several cells, is accounted for algebraically.

In other words, we may be interested in two quantities. One is the amount of water induced from all possible streams into a particular aquifer unit due to pumping from this unit or/and from other units. The other is the amount of water infiltrating from a reach of a stream into the groundwater, due to pumpage from wells located throughout the system.

In both cases, the concept of linear superposition is used, considering these flows as composed of algebraically summed water flows.

As was previously indicated, in this study we are primarily interested in the coupling of a water resources system, consisting of both ground and surface water, with a desired management model. It is possible to extend the algebraic functions relating pumpage from wells to flow between the stream and aquifer to encompass a more complex system. The basis used is that the quantity of water induced from the stream into a unit aquifer (cell) may be approximated by a linear combination of quantities induced from the stream into the particular cell due to pumping at other cells, and of quantities induced due to pumpage from wells belonging to the particular cell.

Let $\phi_r^u(j,n-i+1)$ be the quantity of water induced from the u^{th} stream

into the r^{th} cell during the n^{th} period due to one unit of pumping at the j^{th} well located at the r^{th} cell during the i^{th} time period.

Let $\psi_r^u(\ell,n-i+1)$ be the quantity of water induced from the u^{th} stream into the r^{th} cell during the n^{th} period due to one unit of pumping at the ℓ^{th} cell during the i^{th} time period.

Let $f_r^u(n)$ be the quantity of water induced from the u^{th} stream into the r^{th} cell during the n^{th} time period.

$$f_r^u(n) = \sum_{j=1}^{J_r} \sum_{i=1}^{n} \phi_r^u(j,n-i+1)q_r(j,i) + \sum_{\substack{\ell=1 \\ \ell\neq r}}^{R} \sum_{i=1}^{n} \psi_r^u(\ell,n-i+1)q(\ell,i) + I_r^u \qquad (3.24)$$

where J_r = total number of wells operating in the r^{th} cell.

R = total number of cells $r = 1,\ldots,R$.

$q_r(j,i)$ = quantity of water pumped at the j^{th} well located at the r^{th} cell during the i^{th} time period.

$q(\ell,i)$ = aggregated pumpage at the ℓ^{th} cell area during the i^{th} time period.

I_r^u = the quantity of water induced from the u^{th} stream into the r^{th} cell during one time period with the system in steady state and no pumpage imposed.

The first summation term is the quantity of water induced from the u^{th} stream into the r^{th} cell due to pumpage inside the cell, while the second summation term is the quantity of water induced from the u^{th} stream into the r^{th} cell due to pumping at all other cells.

The quantity of water induced from all streams (reaches) into the r^{th} cell during the n^{th} time period is defined by $f_r^{\bar{U}}(n)$

$$f_r^{\bar{U}}(n) = \sum_{u=1}^{U} f_r^u(n) \qquad (3.25)$$

U = total number of streams (reaches) traversing the area.

The quantity of water induced from the u^{th} stream into the groundwater system is defined by $f^u_{\overline{R}}(n)$

$$f^u_{\overline{R}}(n) = \sum_{r=1}^{R} f^u_r(n) \tag{3.26}$$

The set of linear functions relating quantities of water induced from streams acting as constant head boundaries into aquifers may be performed subject to the availability of the matrices of parameters denoted by ϕ and ψ. The calculation procedure for these parameters is essentially similar to the procedure suggested for the beta calculation, Section 3.2. A digital model of the aquifer system, with the streams acting as constant head boundaries, may be used. The flow from a constant head node in the discretized model to a neighboring node is directly related to the head difference between the nodes. Once the aquifer head distribution is solved for a certain time period, the quantity of water induced from the constant head node into the non-constant head node is easily determined. Thus, we follow the procedure proposed for the beta calculation. A unit pumpage is imposed at the j^{th} well (or the ℓ^{th} cell) during the first time period. The desired parameter is the flow due to that pumpage at time i, from the node belonging to the u^{th} stream into the node belonging to the r^{th} aquifer unit neighboring the stream.

3.5.4 A Case Study for the Applications of the Stream-Aquifer Interflow Functions.

In Section 3.4 the Fairfield-New Baltimore aquifer was introduced as a case study. The Great Miami River traverses the area and in most cases acts as a constant head boundary for the groundwater system.

To illustrate the usefulness of the proposed technique for determining the parameter relating the quantity of water induced from a stream into an aquifer due to pumpage from wells, the multicell model of the Fairfield area is used. Two sets of parameters are calculated.

First, the parameter I^u_r which denotes the quantity of water induced from the u^{th} stream into the r^{th} cell per unit of time in steady state conditions with no imposed pumpage is determined for some u and r defined in the model.

Secondly, the parameter $\psi_r^u(\ell,n-i+1)$ which denotes the quantity of water induced from the u^{th} stream into the r^{th} cell during the n^{th} period due to one unit pumping at the ℓ^{th} cell during the i^{th} period, is determined for some ℓ,r,u.

The multicell model is described in detail in Phase I, Chapter 5. To solve for I_r^u, the model is run over a long time period to provide a steady state head and flow distribution, with no pumpage imposed. Recall that the stream is represented in the model as a group of constant head cells (See Figure 3.1). The river corresponds to cells 10, 11, and 12 which may be referred to as reaches of the stream, $u = 10, 11, 12$. Table 3.5 summarizes the results.

The Fairfield Aquifer Area

I_r^u Values $\left[1000Ft^3/Day\right]$

Reach u	10	10	11	12
Cell r	4	5	5	7
I_r^u	-1210	1230	-860	-120

TABLE 3.5

To solve for $\psi_r^u(\ell,n-i+1)$, a unit of pumpage is imposed on a particular cell, and the simulation model provides the flows from the different streams into the neighboring cells over time, until the steady state is reached again. The differences between these flows and the corresponding I_r^u values are the desired parameters $\psi_r^u(\ell,n-i+1)$. Table 3.6 summarizes the results for $\ell = 4,5$, where the pumpage is imposed during the year $i = 1$.

The Fairfield Aquifer Area

$\psi_r^u(\ell,n)$ Values $\left[1000\ Ft^3/Day\right]$

(One Unit Pumpage imposed on ℓ during the $i = 1$ Period)

u	10		10		11		12	
r	4		5		5		7	
ℓ	4	5	4	5	4	5	4	5
n								
1	557	-	-	190	-	290	-	10
2	52	-	-	120	-	190	-	20
3	5	-	-	30	-	40	-	10
4	-	-	-	10	-	15	-	5

TABLE 3.6

If 1,000 Ft^3/Day is pumped from Cell 5 through the year 1, it will cause the river at Reach 10 to induce 190 Ft^3/Day at year 1 into Cell 5 and 120 Ft^3/Day at year 2. It will also cause the stream at Reach 12 to induce 10,20,10,5 Ft^3/Day through years 1,2,3,4 respectively. However, pumping at Cell 5 may not effect the flow between Reach 10 and Cell 4.

Chapter 4

THE COUPLED RESPONSE OF COMBINED GROUND AND SURFACE WATER SYSTEMS UNDER CHANGES IN STREAM-AQUIFER BOUNDARY CONDITIONS

4.1 General Discussion

In previous sections the development of functions relating drawdown to pumping at wells was discussed. Also functions relating the quantity of water induced from streams into the aquifer and pumpage were formulated. In both types of functions, the water resources system was considered to comprise a groundwater aquifer system traversed by streams, and the streams were considered to act as constant head boundaries over the time period simulated.

Departing from the preceeding discussion, we now wish to investigate the case where, due to activities imposed on the system, some of the boundary conditions assumed initially to hold are subject to some changes at an unknown time within the time period of interest.

4.2 The Time Varying Boundary Conditions.

A detailed description of the water resources system investigated in this study is given in Sections 3.2 and 3.5.2. To complete the mathematical model, initial and boundary conditions are to be defined.

The drawdown at a point $\hat{x} = (x,y)$, $d(\hat{x},t)$ is given by the partial differential equation:

$$\frac{\partial}{\partial x}\left[T(\hat{x},t)\,\frac{\partial}{\partial x}\left[d(\hat{x},t)\right]\right] = S(\hat{x})\frac{\partial}{\partial t}\left[d(\hat{x},t)\right] + \sum_{j=1}^{M} q(\hat{x}_j,t)\,\delta(\hat{x}-\hat{x}_j) \qquad (4.1)$$

Where $T(\hat{x},t)$ and $S(\hat{x})$ are transmissivity and storage coefficient, respectively, $\delta(\hat{x}-\hat{x}_j)$ is a Dirac Delta function, $\hat{x}_j$ indicates the position of the j-th well and $q(\hat{x}_j,t)$ is the instantaneous discharge at the j^{th} well. M is the total number of wells.

Let the no-flow boundary conditions be $\frac{\partial d(\lambda,t)}{\partial n} = 0 \;\forall\, t$,

where λ is a parameter indicating that $\frac{\partial d}{\partial n}$ is evaluated on the boundary (which is irregular in shape). The n is the normal direction and the $\frac{\partial d(\lambda,t)}{\partial n}$ is the gradient of the drawdown for the normal to the boundary, which vanishes under no-flow conditions.

Let the constant head boundary condition be $d(\mu,t) = 0$ for all t. where μ is a parameter indicating that the drawdown d is evaluated along the constant head boundary.

Suppose now that the boundary conditions are not only a function of space but also of time. That is possible due to activities imposed on the water resources system, reflected by the forcing function $\hat{q}(x_j,t)$.

Unless the aquifer is characterized by a very special set of specifications, it is unlikely that a no-flow boundary condition will qualify as time varying. As contrasted with no-flow boundaries, constant head boundary conditions may appear to act as such only under very restricted situations. For a given system the conditions hold if the aquifer water table in the close vicinity of the stream is found to be in a range where the hydraulic stream-aquifer interrelations do define a constant head boundary for the aquifer. However, once the aquifer water level is lowered below that range, the hydraulic interrelations are transformed into flow conditions between the stream and the aquifer. That flow may be considered as a function of the stream only -- its depth, temperature, etc. The aquifer water level is no longer a principal component in determining the infiltration rate.

If the model relates to short-term planning, the fluctuations in the stream water level and other characteristics must be taken into account. However, for a long-term planning, it is sometimes acceptable to consider the average stream conditions. If these conditions are not affected by the aquifer water head or by some activities imposed on the stream as direct pumpage on it, the infiltration rate may be approximated as a constant flow from the stream into the aquifer. This is in a sense the basic assumption underlying the following discussion. The application of these methodologies to be developed to short-term planning is possible, although modeling complications may be induced.

It is therefore useful to assume that between the stream and aquifer, the hydraulic relations may define only one of the two situations, namely constant head boundary conditions or constant flow forcing function. In a more detailed formulation, one may replace the constant infiltration by a function including the stream water level and bed parameters.

Assume a change in the boundary conditions at time $0 < t_1 < T$ to be a constant head changing into constant flow recharging the aquifer along the boundaries denoted by μ. T denotes the time horizon of planning.

The constant flow is defined by $q(\mu,t) = q(\mu)$. This indicates that along the boundaries μ, a constant flow $q(\mu)$ is forcing the system: it is time invariant.

Let the initial conditions for the constant head case be $d(\hat{x},o) = 0$ corresponding to no previous development. The drawdown at any point $\hat{x}$ and time t_1 is

$$d(\hat{x},t_1) = \sum_{j=1}^{M} \int_{o}^{t_1} G_{CH}(\hat{x},\hat{x}_j,t_1-\tau)q(\hat{x}_j,\tau)d\tau \qquad (4.2)$$

where G_{CH} is the Green's function evaluated under constant head boundary conditions (see Appendix A). At time t_1 we assume that the infiltration rate reaches its maximum value. In the following period the stream-aquifer relations are based on constant flow recharging the aquifer, at a rate equal to the maximum infiltration rate.

The system is now defined by different conditions. The initial conditions for the constant flow situation are defined by equation (4-2). The boundary conditions do not include the constant head boundaries which were transformed into a constant flow forcing function along the stream boundaries. The drawdown at any point in time $t>t_1$ is $d(\hat{x},t)$:

$$d(\hat{x},t) = \sum_{j=1}^{M} \int_{t_1}^{t} G_{CF}(\hat{x},\hat{x}_j,t-\tau)q(\hat{x}_j,\tau)d\tau \qquad t>t_1 \qquad (4.3)$$

where G_{CF} is the green's function evaluated under the new initial conditions (4-2) and boundary conditions including the constant flow conditions.

In order that equation (4-3) be solvable, the time of boundary condition change t_1 has to be determined. For a given pattern of $q(\hat{x}_j,t)$, on the interval $t\varepsilon(o,t_1)$ the Greens function G_{CH} holds. t_1 is determined by the relation (4-4).

$$Q^{\mu}_{INF}(t_1) = \bar{Q}^{\mu}_{INF} \tag{4.4}$$

where $\bar{Q}^{\mu}_{INF}$ is the maximum infiltration rate along the stream bed. $Q^{\mu}_{INF}(t)$ is the quantity of water induced from the stream into the aquifer. Equation (4-4) states that t_1 is the time when this quantity reaches its maximum infiltration rate $\bar{Q}^{\mu}_{INF}$.

We assume that during the time interval (O,t_1) the infiltration rate is changing with time, due to changes in the groundwater. For simplicity the rate can be represented as unchanged within any of the unit time periods. For a developing region it is acceptable to consider the pumping patterns $q(j,i)$ to increase over time, such that once $Q^{\mu}_{INF}(t)$ reaches the value of $\bar{Q}^{\mu}_{INF}$ the constant head boundary conditions may not be incured again.

A principal difficulty in solving for the drawdown through the aquifer after the time t_1 is due to the initial conditions (4-2), resulting from the system behavior prior to t_1. The results of Chapter 3 are used to handle this difficulty. Recall that the constant head boundary conditions imposed by the stream on the aquifer are affecting the groundwater system as if a forcing flow is induced along the stream such that the water level remains steady on the stream line. This implies that the quantity of water induced from the stream into the aquifer is essentially balancing the quantity of water leaving the aquifer section underlying the stream. In other words, it is possible to replace the constant head boundary conditions in the model by forcing flow into the aquifer along the stream, provided that the rate is determined by the equivalent constant head situation.

For the time interval (O,t_1), the drawdown at any point $\hat{x}$, within defined boundaries including the constant head boundary conditions imposed by the stream, is given by (4.2) as:

$$d(\hat{x},t)=\sum_{j=1}^{M} \int_{o}^{t} G_{CH}(\hat{x},\hat{x}_j,\ t-\tau)\ q\ (\hat{x}_j,t)\ d\tau, t < t_1. \tag{4.2}$$

In section 3.5.2 we show that the quantity of water $F_R(t)$ removed from aquifer storage over time t is:

$$F_R(t) = \sum_{j=1}^{M} \int_o^t \left[\iint_{\hat{x}} S(\hat{x}) G_{CH}(\hat{x},\hat{x}_j,t-\tau)\, d\hat{x} \right] q(\hat{x}_j,\tau)\, d\tau \quad t<t_1 \qquad (4.5)$$

Now, the quantity of water $Q^{\mu}_{INF}(t)$ removed from stream over time t in the interval $(0,t_1)$ by interacting with the aquifer along the line indicated by μ is:

$$Q^{\mu}_{INF}(t) = \sum_{j=1}^{M} \int_o^t \left\{ 1 - \left[\int_{\hat{x}} S(\hat{x})\, G_{CH}(\hat{x},\hat{x}_j,t-\tau)\, d\hat{x} \right] \right\} q(\hat{x}_j,\tau) d\tau \qquad (4.6)$$

(We assume here that water is available only from storage and the stream)

The function $Q^{\mu}_{INF}(t)$ may be viewed as the input function imposed on the aquifer due to the stream-aquifer interaction over the time interval $(0,t_1)$.

For $t>t_1$, $Q^{\mu}_{INF}(t) = \bar{Q}^{\mu}_{INF}$ = constant,
where $\bar{Q}^{\mu}_{INF}$ denotes the maximum infiltration allowable along the stream bed restricted by the physical characteristics of the system.

The drawdown at a point $\hat{x}$ in time $t>t_1$ as given by (4-3):

$$d(\hat{x},t) = \sum_{j=1}^{M} \int_{t_1}^t G_{CF}(\hat{x},\hat{x}_j,t-\tau)\, q(\hat{x}_j,\tau)\, d\tau \quad t \in [t_1,T] \qquad (4.3)$$

We define $G_T(\hat{x},\hat{x}_j,t-\tau)$, a Green's function which solves the aquifer partial differential equation under initial conditions $d(\hat{x},o)= 0$ and boundary conditions including the stream-aquifer interactions. These interactions will therefore act as a forcing function of magnitude $Q^{\mu}_{INF}(t)$, $t\in [o,t_1]$, and $\bar{Q}^{\mu}_{INF}$, $t> t_1$.

The drawdown $d(\hat{x},t)$ at any point $\hat{x}$ in time $o\leq t\leq T$ is

$$d(\hat{x},t) = \sum_{j=1}^{M} \left[\int_o^t G_T(\hat{x},\hat{x}_j,t-\tau) q(\hat{x}_j,\tau)\, d\tau \right]$$

$$\int_o^{t_1} G_T(\hat{x},\mu,t-\tau)\, Q^{\mu}_{INF}(\tau)\, d\tau \qquad (4.7)$$

$$\int_{t_1}^t G_T(\hat{x},\mu,t-\tau)\, \bar{Q}^{\mu}_{INF}\, d\tau$$

Where

t_1 is subject to $Q^{u}_{INF}(t_1) = \bar{Q}^{u}_{INF}$ (4.8)

And

$$Q^{u}_{INF}(t) = \sum_{j=1}^{M} \int_{o}^{t} \left\{ 1 - \left[\int\int_{\hat{x}} S(\hat{x})\, G_{CH}(\hat{x},\hat{x}_j, t-\tau)\, d\hat{x} \right] \right\} q(\hat{x}_j,\tau)\, d\tau \qquad (4.9)$$

The first term in equation (4-7) is the drawdown at point $\hat{x}$ due to pumpage at all wells over time $o \leq t \leq T$. The second term is the drawdown due to water induced from the stream acting as a constant head boundary for the aquifer over time $o \leq t \leq t_1$. The third term is the drawdown caused by a constant infiltration rate from the stream into the aquifer over time $t > t_1$, when the stream no longer acts as constant head boundary. The last two terms are negative as they impose a reduction of the drawdown.

One may notice that in our formulation the stream is taken as a single water point source while in reality the stream-aquifer interaction is distributed along the stream line. Taking into account the distributed nature of the phenomenon might make it necessary to consider the possibility that, while in one section of the stream at one time constant head boundary conditions do not exist, other sections of the stream are still acting as constant head boundaries.

For such a situation the determination of the time of change t_1 becomes the much more complicated process of determining a vector of interrelating change times.

Let n_u indicate the final time period of the u-th stream acting as a constant head boundary. Recall that the separation of a stream-network into multistream units is a procedure unrelated to the multicell topology. Hence, we define a stream reach u to comprise a unit stream which may be regarded as a single subsystem playing the role of constant head boundary for the groundwater system during the time periods $n=1,....n_u$ and constant inflow during the time periods $n= n_u+1,....N$.

It is now possible to determine the drawdown at any cell in the multicell model for the case where the u-th stream relations to the aquifer are

subject to change after n_u time periods, from a constant head boundary into a constant flow forcing function.

Under the assumption that the quantity of water pumped may vary from time step to time step, the drawdown at the ℓ-th cell in the multicell model at the end of n time periods is (see section 3.3):

$$d(\ell,n) = \sum_{r=1}^{R} \sum_{i=1}^{n} \gamma(\ell,r,n-i+1)\, q(r,i) + \sum_{i=1}^{n_u} \gamma(\ell,u,n-i+1)\, f_\ell^u(i) + \sum_{i=n_u+1}^{n} \gamma(\ell,u,n-i+1)\, \bar{Q}^u_{INF,\ell} \qquad (4.10)$$

Where

$$f_\ell^u(i) = \sum_{j=1}^{J\ell} \sum_{p=1}^{i} \phi_\ell^u(j,i-p+1)\, q_\ell(i,p) + \sum_{\substack{r=1 \\ r\neq\ell}}^{R} \sum_{p=1}^{i} \psi_\ell^u(r,i-p+1)\, q(r,p) \qquad (4.11)$$

n_u is subject to $f_\ell^u(n_u) = \bar{Q}^u_{INF,\ell}$ (4.12)

and $Q^u_{INF,\ell}$ is the maximum infiltrating rate from the u-th stream into the ℓ-th cell.

The gamma parameters in equation (4-10) correspond to the multicell model where the u-th stream does not act as a constant head boundary, and hence does not appear in the multicell structure and model formulation. The parameters ϕ and ψ in equation (4-11) correspond to the multicell and particular cell in which the u-th stream plays the role of a constant head boundary condition.

The disadvantage of needing to formulate two different multicell models for such a case is even greater when the number of candidate streams for a change from constant head to constant flow boundary increases. In that case we must examine all combinations of possible models where different streams act

as constant head boundaries. However, adopting the multicell approach reduces this problem to an acceptable size, and in many cases where the pumpage is concentrated at a particular single cell, the proposed procedure is very acceptable. For such a case, the drawdown at any particular point in the groundwater system may be determined via the same procedure as proposed in section 3.3, with equations (4.10), (4.11), (4.12) representing the modified multicell model.

Chapter 5

A PREDICTIVE MODEL

5.1 General Discussion

In this chapter we will consider a water resources system encompassing both ground and surface water use by several users, where there is no coordination mechanism within the region. Each user operates his own water supply system using wells and artificial recharge facilities, and is concerned only with his own goals.

Two basic structures are considered. In the first case the water demand is given for all users during the planning period, and is thus independent of the cost of making the water available for use. Also for this structure we assume that artificial recharge plans by each user are not control variables as these plans are independent of the pumping plans and may be considered given for the entire planning period. In the second case water demand is a function of water use, and the artificial recharge plan must be determined for optimal operation. This case is referred to as the elastic case. The basic structure of the physical system for both cases is identical.

To conclude our general discussion, it must be understood that the developments in this chapter are the first step in constructing this study's modeling procedures for regional water resources management. The general framework of this and the following chapter is briefly illustrated in Figure 5.1.

5.2 The Basic Regional System

In the previous discussion, the case with no coordination between users was introduced. In particular, a basin comprising aquifers traversed by streams is considered. Users throughout the basin pump water from aquifers by means of operating wells. Each user's desire for water is primarily governed by economics, but he may also consider the stream water response, e.g. water level and quality in the vicinity of his location. Except for artificial recharge purposes, the direct use of surface water will not be considered at this stage.

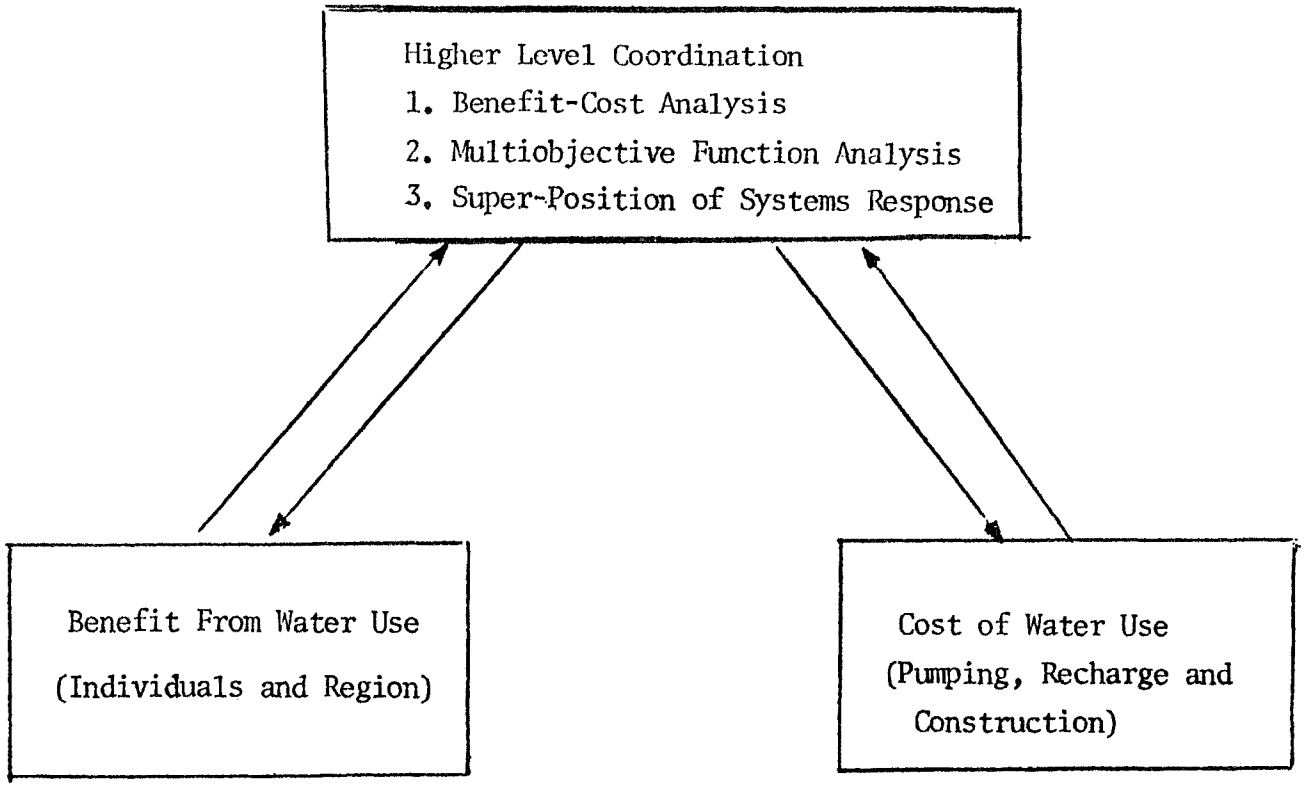

FIGURE 5.1

A Framework of the Study's Mcdeling Approach

for Regional Water Resources Management

The stochastic nature of stream flows, natural recharge to the groundwater and other such aspects affecting the water balance in the system, may play an essential role in a real system. The preliminary development here, however, is deterministic, in order to focus on the modeling procedures.

Extension to a stochastic real system will be described later. Similarily, later developments will extend the system to include a surface water source, which will directly compete with groundwater supply in meeting demand.

We assume that for each single user, there is one aquifer cell from which he pumps his water needs by operating one or more wells. A single cell may underlie a number of stream reaches. Note that this definition of an aquifer cell is not restricted to geological or hydrological boundaries, though it may be subject to geographical, legal or political bounds.

If a user operates artificial recharge facilities, these are considered aggregated at a single point inside his defined area. Water is transferred to this point from the different streams according to the recharge plan.

For the case of inelastic water demand, the economic criterion is the gross cost of water supply. Each user attempts to minimize the capital, operational, and maintenance and replacement cost associated with water use and artificial replenishment.

With water demand as a function of the water price, the economic criterion is the net benefit obtained from water use.

The method of model superposition applied to either case may show a real advantage in the formulation process as well as in the solution strategy. The single optimization problems conducted by each of the users are coupled to each other through the physical system. The proposed methodology enables the decoupling of these programs. A general responsive model provides each of the users with the following information:

I) Water levels at different operating wells during the time horizon.
II) The expected time for drawdown at some wells to exceed casing and screening designs.

III) The quantity of water induced from the stream into aquifer in the vicinity of his operating wells.

This information may cause the user to change his operational and design plans, either to reduce per unit water cost, or to increase his net benefit.

These revised plans are not expected to effect total demand patterns for the inelastic case. They may, however, effect the following:

I) The operational plans of particular wells. Quantities pumped from some wells may be transferred to other wells within the aquifer cell.

II) The design plans. He will redesign drilling of wells and pipeline construction based upon the expected water levels in the aquifer and the stream as determined by the responsive model.

If demand is a function of water price, the total pumpage pattern and recharge plans of each user may also be subject to changes. In the following sections, these two programs will be formulated in detail, i.e. for

I) Inelastic demand

II) Elastic demand.

5.3 Model Formulation

5.3.1 Inelastic Demand

There are L users in the region. To each user corresponds an aquifer cell, and the ℓ-th user has m_ℓ wells located at the ℓ-th cell. Each user $\ell \in L$ attempts to minimize his own pumping cost Z_ℓ:

$$\min Z_\ell = \sum_{n=1}^{T} (1+r)^{-n} \left[C_\ell(n) + \sum_{k_\ell=1}^{m_\ell} P_\ell(k_\ell) q_\ell(k_\ell,n) (H_\ell(k_\ell) + d_\ell(k_\ell,n) + D(\ell,n)) \right] \tag{5.1}$$

T	is the number of time periods in the design horizon.
r	is the interest rate
m_ℓ	is the number of wells located at the ℓ-th cell and operated by the ℓ-th user
$C_\ell(n)$	is the construction cost to the ℓ-th user at the n-th period according to his particular plans
$q_\ell(k_\ell,n)$	is the quantity of water pumped from the k_ℓ-th well during the n-th period

$P_\ell(k_\ell)$ is the pumping cost per acre-ft for the k_ℓ-th well

$H_\ell(k_\ell)$ is the lift under steady state conditions at the k_ℓ-th well

$d_\ell(k_\ell,n)$ is the drawdown in the k_ℓ-th well at the end of the n-th period due to the aggregated pumping and recharge in the ℓ-th cell

$D(\ell,n)$ is the drawdown in the ℓ-th cell at the end of the n-th time period due to the aggregated pumpage and recharge in all other cells (i.e. by all other users) in the region.

$d_\ell(k_\ell,n)$ is given by:

$$d_\ell(k_\ell,n) = \sum_{j=1}^{m_\ell} \sum_{i=1}^{n} \beta_\ell(k_\ell,j,n-i+1)q_\ell(j,i) - \sum_{i=1}^{n}\left[\beta_\ell(k_\ell,v_\ell,n-i+1) \cdot \sum_{u=1}^{u_\ell} v_\ell(u,i)\right] \quad (5.2)$$

where $\beta_\ell(k_\ell,j,n-i+1)$ is the algebraic technological term relating the drawdown at the k_ℓ-th well to the pumping of one unit of water from the j-th well during the i-th period. Both k_ℓ and j are located at the ℓ-th cell. The second term on the right of (5.2) represents the negative drawdown at well k_ℓ due to artificial recharge at point v_ℓ, where $v_\ell(u,i)$ is the quantity of water from the u-th stream used for artificial recharge at the ℓ-th recharge facility during period i, and there are u_ℓ reaches of streams traversing the ℓ-th cell area.

$D(\ell,n)$ is given by:

$$D(\ell,n) = \sum_{\substack{r=1 \\ r\neq\ell}}^{L} \sum_{i=1}^{n} \gamma(\ell,r,n-i+1)q_N(r,i) \quad (5.3)$$

where $\gamma(\ell,r,n-i+1)$ is the algebraic technological term relating the average drawdown at the ℓ-th cell to aggregated pumping of one unit of water at the r-th cell during the i-th period.

$q_N(r,i)$ is the net quantity of water pumped from the r-th cell by the r-th user during the i-th period.

The ℓ-th user is interested in minimizing expression (5.1) subject to such constraints as:

I) Water requirements must be met

$$\sum_{k_\ell=1}^{m_\ell} q_\ell(k_\ell,n) \geq R_\ell(n) \qquad n=1,\ldots,T \tag{5.4}$$

II) Drawdowns must not exceed casing and screening designs

$$d_\ell(k_\ell,n) + D(\ell,n) \leq d_\ell max(k_\ell) \qquad \begin{matrix} n=1,\ldots,T \\ k_\ell=1,\ldots,m_\ell \end{matrix} \tag{5.5}$$

III) Pumping capacity must not be exceeded

$$q_\ell(k_\ell,n) \leq Q_\ell max(k_\ell) \qquad \begin{matrix} n=1,\ldots,T \\ k_\ell=1,\ldots,m_\ell \end{matrix} \tag{5.6}$$

IV) Water Balance in the stream must be maintained

$$\sum_{\text{all } \ell} \left[v_\ell(u,n) + f_\ell^u(n) \right] \leq B_\ell(u,n) \qquad \begin{matrix} n=1,\ldots,T \\ u=1,\ldots,u_\ell \end{matrix} \tag{5.7}$$

$R_\ell(n)$ is the water requirement for the ℓ-th user in the n-th time period (a fixed quantity for each n).

$d_\ell(k_\ell,n)$ and $D(\ell,n)$ are given by (5.2) and (5.3), respectively. $d_\ell max(k_\ell)$ is the maximum drawdown allowed for the k_ℓ-th well located at the ℓ-th cell, which must not be exceeded because of casing and screening design.

$Q_\ell max(k_\ell)$ is the designed upper limit on the quantity of water pumped from the k_ℓ-th well.

$B_\ell(u,n)$ is the upper limit of quantity of water to be removed from the u-th stream for natural and artificial recharge.

$f_\ell^u(n)$ is the quantity of water induced from the u-th stream into the ℓ-th cell during the n-th period:

$$f_\ell^u(n) = \sum_{k_\ell=1}^{m_\ell} \sum_{i=1}^{n} \phi_\ell^u(k_\ell, n-i+1) q_\ell(k_\ell, i) + \sum_{\substack{r=1 \\ r\neq\ell}}^{L} \sum_{i=1}^{n} \psi_\ell^u(r, n-i+1) q_N(r,i) + I_\ell^u \tag{5.8}$$

$\phi_\ell^u(k_\ell, n-i+1)$ is the quantity of water induced from the u-th stream into the ℓ-th cell during the n-th period due to one unit of pumping at the k_ℓ-th well during the i-th period.

$q_\ell(k_\ell, i)$ is the quantity of water pumped from the k_ℓ-th well during the i-th period.

$\psi_\ell^u(r, n-i+1)$ is the quantity of water induced from the u-th stream into the ℓ-th cell during the n-th period due to one unit of pumping at the r-th cell during the i-th period.

$q_N(r,i)$ is the net quantity of water pumped from the r-th cell during the i-th period.

I_ℓ^u is the quantity of water induced from the u-th stream into the ℓ-th cell during one time period with no imposed pumpage and the system in steady state.

$v_\ell(u,i)$ is the quantity of water from the u-th stream used for recharge at v_ℓ during period i.

Expressions (5.1) through (5.8) constitute a quadratic program. Corresponding to each particular user, there are L such disjoint programs within the region. Each user has to consider two types of decisions:

1. Design oriented (long term planning). These decisions include the water supply system development, e.g., the location of new wells or the establishment of a water distribution system. Associated with this kind of decision are the cost functions for construction $C_\ell(n)$ and operation $P_\ell(n)$, as well as also the limiting design values of $d_{\ell max}(k)$ (the maximum allowable drawdown) and $Q_{\ell max}(k_e)$ (the maximum pumpage capacity of wells).

2. Operation oriented (short-term planning). These decisions result from the solution of the optimization (quadratic) program and include the pumping patterns $q_\ell(k_\ell,n)$ which comprise the pumping plan.

A special class of decisions concern water quality and legal restrictions. The term $B_\ell(u,n)$ represents the desire to not exceed some limit of stream flow balance by maintaining the infiltration rate. Each particular user must decide on the desired level of $B_\ell(u,n)$ within the restrictions imposed by law. If this value limits the availability of water, the user may possibly decide to increase it, even if some additional cost (fine) is imposed.

Each individual program can be solved in isolation. The L programs are coupled through the physical system responses, including the $D(\ell,n)$ and $f_\ell^u(n)$ functions relating the system response on the ℓ-th user from pumpage imposed in other parts of the hydrologic system by other users. Nevertheless, in this case where the water demand is not a function of the water price, the quantity of water pumped from the ℓ-th cell during the n-th period, $q(\ell,n)$ is essentially identical to the water requirement for the ℓ-th user in the n-th period, $R_\ell(n)$:

$$q(\ell,n) = R_\ell(n) \quad \begin{matrix}\ell=1,\ldots,L \\ n=1,\ldots,T\end{matrix} \tag{5.9}$$

where $q(\ell,n)$ is the term for the gross pumpage at the ℓ-th cell during the n-th period.

The coupling function

$$D(\ell,n) = \sum_{\substack{r=1 \\ r\neq\ell}}^{L} \sum_{i=1}^{n} \gamma(\ell,r,n-i+1) q_N(r,i) = \sum_{\substack{r=1 \\ r\neq\ell}}^{L} \sum_{i=1}^{n} \gamma(\ell,r,n-i+1)\left[R_r(n)-v_r(n)\right] \tag{5.10}$$

$R_r(n) - v_r(n)$ is a given term for all r and n. As such, $D(\ell,n)$ once determined, represent for the ℓ-th user the aggregated drawdown at his cell due to all other user's activities. It may be considered as unvarying information, unrelated to his (or other's)decisions with respect to operational or design plans.

The coupling function $f_\ell^u(n)$ is given in (5.8). This can be reformulated as:

$$f_\ell^u(n) = {}^u_{fP_\ell}(n) + {}^u_{fR_\ell}(n) + I_\ell^u \tag{5.11}$$

where

$${}^u_{fP_\ell}(n) = \sum_{k_\ell=1}^{m_\ell} \sum_{i=1}^{n} \phi_\ell^u(k_\ell,n-i-1)\, q_\ell(k_\ell,i) - \sum_{i=1}^{n} \phi_\ell^u(v_\ell,n-i+1) \cdot \sum_{u=1}^{u_\ell} v_\ell(u,i) \tag{5.12}$$

and

$${}^u_{fR_\ell}(n) = \sum_{\substack{r=1 \\ r\neq\ell}}^{L} \sum_{i=1}^{n} \psi_\ell^u(r,n-i+1) q_N(r,i) \tag{5.13}$$

${}^u_{fP_\ell}(n)$ and ${}^u_{fR_\ell}(n)$ are the quantity of water induced from the u-th stream into the ℓ-th cell during the n-th period due to net pumping from wells located inside the ℓ-th cell and pumping from other cells, respectively.

Only the term ${}^u_{fR_\ell}(n)$ couples the particular user problem with the rest of the system activities. However, once ${}^u_{fR_\ell}(n)$ is determined for a given set of water requirements for all the users, it is considered unvarying, and so the ℓ-th user problem is completely uncoupled from the rest and can be separately solved.

The procedure for simulating the future developments in the water resources system where no coordination exists between users is summarized in Figure 5.2. Notice again that it is assumed that the water demand is independent of the water utilization price.

5.3.2 Elastic Demand

Consider the system described previously, but with water requirements now assumed flexible and subject to the cost of water availability. Each user $\ell \in L$ maximizes his own net revenues NR_ℓ:

$$\max NR_\ell = \sum_{n=1}^{T} \{(1+r)^{-n} \left[B_\ell(n) \cdot \sum_{k_\ell=1}^{m_\ell} q_\ell(k_\ell,n) - V_\ell(n) \cdot \sum_{u=1}^{u_\ell} v_\ell(u,n) \right]\} - Z_\ell \tag{5.14}$$

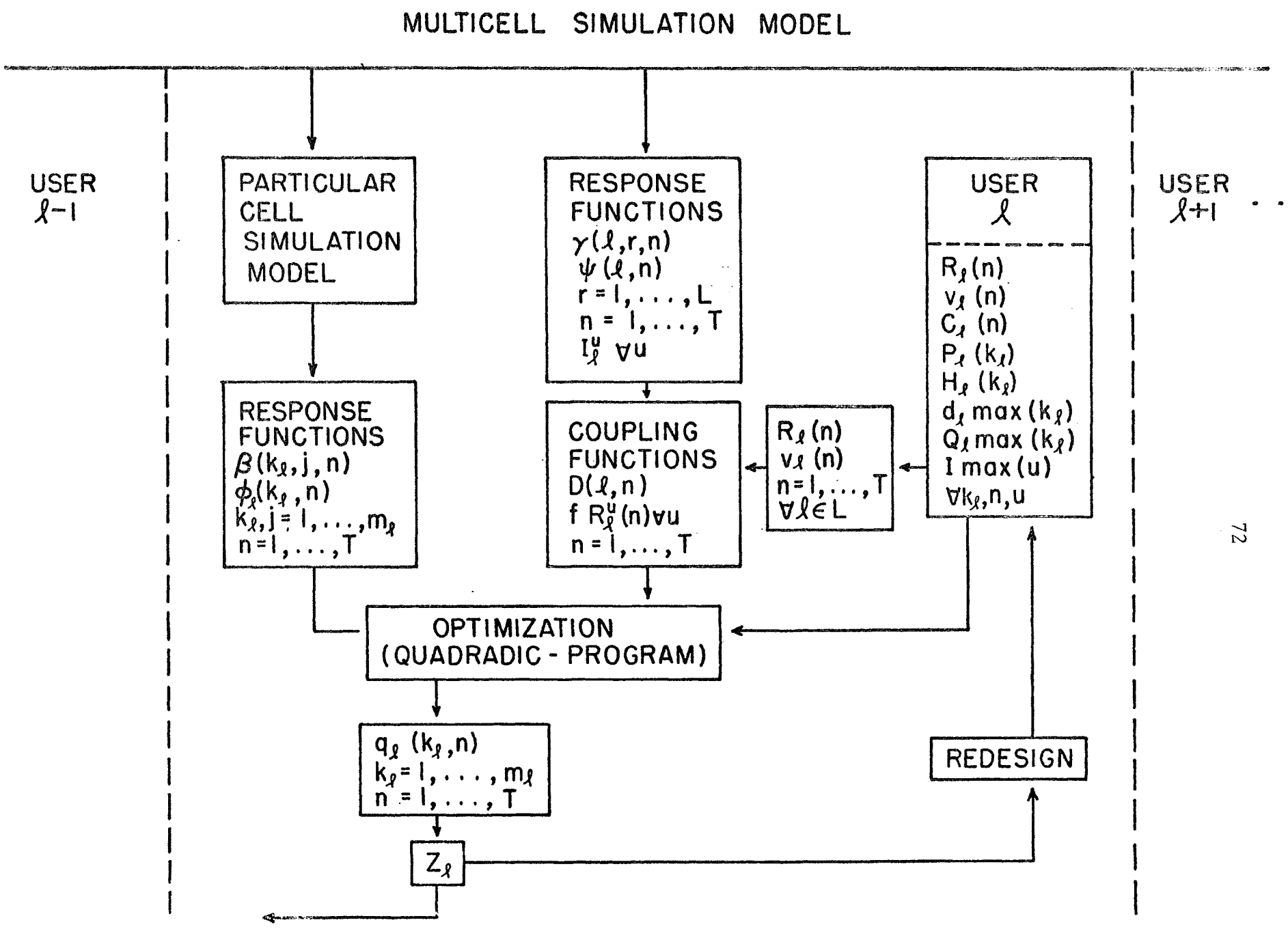

Figure 5.2: A simulation chart of future developments in water resources system, with optional considerations taken by each individual user and no coordination between users.

$B_\ell(n)$ is the return per acre-ft of water supply for the ℓ-th user during the n-th period. $B_\ell(n)$ can be a constant relating only to ℓ and n, or a function of $\sum_{k=1}^{m_\ell} q_\ell(k_\ell,n)$, the total water supply to user ℓ during period n.

$V_\ell(n)$ is the operating cost of recharge per acre-ft in the ℓ-th area during the n-th period.

Z_ℓ is the cost function:

$$Z_\ell = \sum_{n=1}^{T} (1+r)^{-n}\left[C_\ell(n) + \sum_{k_\ell=1}^{m_\ell} P_\ell(k_\ell)q_\ell(k_\ell,n)(H_\ell(k_\ell) + d_\ell(k_\ell,n)+D(\ell,n))\right] \quad (5.15)$$

$C_\ell(n)$ is the constructing cost to the ℓ-th user at the n-th period, according to his particular plans. $q_\ell(k_\ell,n)$ is the quantity of water pumped from the k_ℓ-th well during the n-th period.

$P_\ell(k_\ell)$ is the pumping cost per acre-ft-ft for the k_ℓ-th well.

$H_\ell(k_\ell)$ is the lift under steady state conditions at the k_ℓ-th well.

$d_\ell(k_\ell,n)$ is the drawdown in the k_ℓ-th well at the end of the n-th period due to aggregated pumpage and recharge in the ℓ-th cell.

$D(\ell,n)$ is the drawdown in the ℓ-th cell at the end of the n-th time period due to aggregated pumpage and recharge in all other cells by other users in the region.

$d_\ell(k_\ell,n)$ is given by:

$$d_\ell(k_\ell,n) = \sum_{j=1}^{m_\ell}\sum_{i=1}^{n} \beta_\ell(k_\ell,j,n-i+1)q_\ell(j,i) - \sum_{i=1}^{n}\left[\beta_\ell(k_\ell,v_\ell,n-i-1)\cdot\sum_{u=1}^{u_\ell} v_\ell(u,i)\right] \quad (5.16)$$

where $\beta_\ell(k_\ell,j,n-i-1)$ is the algebraic technological term relating the drawdown at the k_ℓ-th well to the pumping of one unit of water from the j-th well during the i-th period. Both k_ℓ and j are located at the ℓ-th cell. The second term on the right of (5.16) represents the negative drawdown at well k_ℓ due to artificial recharge at point v_ℓ.

$v_{\ell}(u,i)$ is the quantity of water from the u-th stream used for artificial recharge at the ℓ-th recharge facility during period i, and there are u_{ℓ} reaches of streams traversing the ℓ-th cell area.

$D(\ell,n)$ is given by:

$$D(\ell,n) = \sum_{\substack{n=1 \\ r\neq\ell}}^{L} \sum_{i=1}^{n} \gamma(\ell,r,n-i+1)\left(q(r,i) - \sum_{u=1}^{u_r} v_r(u,i)\right) \tag{5.17}$$

where $\gamma(\ell,r,n-i+1)$ is the algebraic technological term relating the average drawdown at the ℓ-th cell to aggregated pumping of one unit of water at the r-th cell, during the i-th period. $q(r,i)$ is the quantity of water pumped from the r-th cell by the r-th user during the i-th period.

The ℓ-th user is interested in maximizing NR_{ℓ}, subject to such constraints as:

I) Non decreasing water supply

$$\sum_{k_{\ell}=1}^{m_{\ell}} q_{\ell}(k_{\ell},n) \leq \sum_{k_{\ell}=1}^{m_{\ell}} q_{\ell}(k_{\ell},n+1) \qquad n=1,\ldots,T-1 \tag{5.18}$$

II) Drawdowns must not exceed casing and screening designs

$$d_{\ell}(k_{\ell},n) + D(\ell,n) \leq d_{\ell max}(k_{\ell}) \qquad \begin{matrix} n=1,\ldots,T \\ k_{\ell}=1,\ldots,m_{\ell} \end{matrix} \tag{5.19}$$

III) Pumping capacity must not be exceeded

$$q_{\ell}(k_{\ell},n) \leq Q_{\ell max}(k_{\ell}) \qquad \begin{matrix} n=1,\ldots,T \\ k_{\ell}=1,\ldots,m_{\ell} \end{matrix} \tag{5.20}$$

IV) Water balance in the stream must be maintained

$$\sum_{\substack{\text{all} \\ \ell}} \left[v_{\ell}(u,n) + f_{\ell}^{u}(n)\right] \leq B_{\ell}(u,n) \qquad \begin{matrix} n=1,\ldots,T \\ u=1,\ldots,u_{\ell} \end{matrix} \tag{5.21}$$

V) Recharge facilities capacity must not be exceeded

$$\sum_{u=1}^{u_{\ell}} v_{\ell}(u,n) \leq v_{\ell max}(n) \qquad n=1,\ldots,T \tag{5.22}$$

$v_{\ell max}(n)$ is the designed upper limit on the quantity of water to be artificially recharged in the ℓ-th cell recharge facilities during period n.

Maximizing NR_ℓ subject to these constraints constitutes a quadratic program. Again there are L such disjoint programs in the region, in this case including an additional decision for each user regarding the quantity of water from different reaches to be transferred for artificial recharge, $v_\ell(u,n)$. Also (5-4), where water requirements impose explicit restrictions on the decision process, is removed.

Inherent in the formulation is the user's need to consider his total pumpage and recharge plans under the benefit-cost analysis where water cost may play an essential role in his particular optimization process.

In the inelastic demand case, the assumption that water requirements for each user were given for all time periods permitted the decoupling of the L optimization programs. This procedure is not applicable to the elastic case because the coupling functions $D(\ell,n)$ - (5.17) and $f_\ell^u(n)$ - (5.8) cannot be solved independently of the particular optimization programs. The quantities $q(r,n)$ and $v_r(n)$ are controlled by each user's individual plans, and these are interdependent. Again note that we are dealing with a structure where no coordination exists and hence no mechanism is available to achieve an 'overall optimum'. The procedure suggested is quite different, requiring an iterative stragegy combining two stages. One stage comprises the L disjoint optimization programs. The other computes the coupling functions associated with the activities resulting from the previous stage. For each iteration, a new set of the coupling functions is determined, and the iterations proceed until no further change in each user's decision variables is indicated. This result gives the activities and the system response in the region for the planning period, allowing the prediction of performance criterion for water resources development in that region under the existing structure.

The procedure is summarized in the following flowchart.

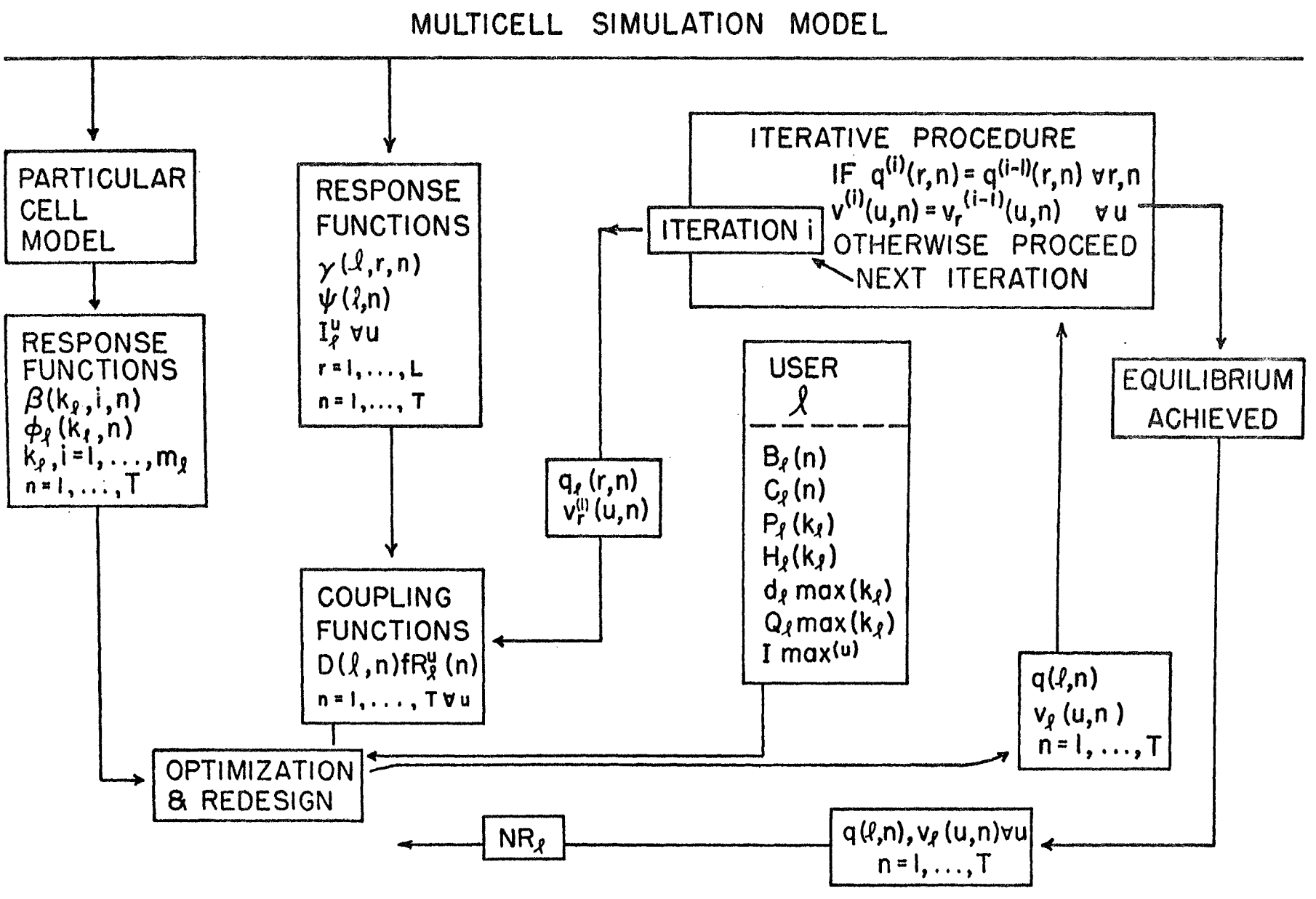

Figure 5.3: A simulating chart of future developments in a water resources system with optimal considerations taken by each individual user, and no coordination between users - The Elastic Demand Case.

Chapter 6

A TAX-QUOTE MODEL IN A MULTICELL-MULTISTREAM SYSTEM

6.1 General Discussion

The objective of the following discussion is to clarify and verify the application of decomposed water resources response functions in the formulation and solution stages of a management model.

Many different management schemes have been utilized in the literature to formulate management models in water resources. Considering a groundwater system traversed by a stream-network, the management mechanism suggested by T. Maddock and Y.Y. Haimes [Maddock and Haimes, 1974] is adopted here. This method of management analysis is presented in detail within a subsequent portion of this report.

In particular, the tax-quota management scheme of Maddock and Haimes can be applied with only minor changes to the water resources system defined in our previous development. The water system in the original study comprised a single aquifer (dry alkaline valley), assuming that no other water sources existed in the region. The mathematical model used for simulating the aquifer is a linear groundwater model in a compact form. Since individual decisions are made for pumping patterns the management model formulation was forced to decompose the decision-making process. However, the physical system model representation (resulting from a compact scheme - the single simulation program) causes each user to have to consider the detailed pumping policy of all the other users. As a result, the management model formulation of the original study requires a great deal of data and computer storage, either of which is not always available. Also, when applied to a real system the modeling efforts are expected to be very difficult.

In the present study we propose the extension of the original approach in two directions:

I) To consider a more complex water resources system comprising multi-aquifer cells traversed by a multi-stream network with artificial recharge and water import options, and a regional performance criterion applied to ground and surface water measurements;

II) To apply the modeling procedure (developed in this study) to the physical system, including the decomposed formulation of the response technological functions.

The management model formulation is expected to be much simplified. The decomposition of the decision making process is followed by a suitable representation of the decomposed physical system response, which can be easily coupled with the management model formulation.

6.2 Model Formulation

There are L users in the region. To each user corresponds an aquifer cell, and the ℓ-th user has m_ℓ wells located at the ℓ-th cell. Each user $\ell \in L$ maximizes his own net revenues NR_ℓ:

$$\max NR_\ell = \sum_{n=1}^{T} \{(1+\rho)^{-n}\Big[B_\ell(n) \cdot \sum_{k_\ell=1}^{m_\ell} q_\ell(k_\ell,n) - \sum_{u=1}^{u_\ell} V_\ell(u) \cdot v_\ell(u,n) - IM_\ell(n) \cdot QI_\ell(n)\Big]\} - Z_\ell \qquad (6.1)$$

T is the number of time periods that comprise the design horizon.

ρ is the interest rate.

m_ℓ is the number of wells located at the ℓ-th cell and operated by the ℓ-th user.

$B_\ell(n)$ is the return per acre-ft of water supply for the ℓ-th user during the n-th period. $B_\ell(n)$ can be a constant relating only to ℓ and n, or a function of $\sum_{k_\ell=1}^{m_\ell} q_\ell(k_\ell,n)$, the total water supply to user ℓ during period n.

$q_\ell(k_\ell,n)$ is the quantity of water pumped from the k_ℓ-th well during the n-th period.

$V_\ell(u)$ is the operating cost of recharge per acre-ft in the ℓ-th area with water from the u-th stream.

$v_\ell(u,n)$ is the quantity of water from the u-th stream used for artificial recharge at the ℓ-th recharge facility during period n, and there are u_ℓ reaches of streams traversing the ℓ-th cell area.

$IM_\ell(n)$ is the cost per acre-ft of water imported into the ℓ-th area during the n-th period.

$QI_\ell(n)$ is the quantity of water imported into the ℓ-th area during the n-th period for direct use by the ℓ-th user.

Z_ℓ is the cost function:

$$Z_\ell = \sum_{n=1}^{\bar{T}} (1+\rho)^{-n} \left[C_\ell(n) + \sum_{k_\ell=1}^{m_\ell} P_\ell(k_\ell) q_\ell(k_\ell,n) (H_\ell(k_\ell) + d_\ell(k_\ell,n) + D(\ell,n)) \right] \quad (6.2)$$

$C_\ell(n)$ is the constructing cost to the ℓ-th user at the n-th period according to his particular plans.

$P_\ell(k_\ell)$ is the pumping cost per acre-ft-ft for the k_ℓ-th well.

$H_\ell(k_\ell)$ is the lift under steady state conditions at the k_ℓ-th well.

$d_\ell(k_\ell,n)$ is the drawdown in the k_ℓ-th well at the end of the n-th period due to the aggregated pumpage and recharge in the ℓ-th cell.

$D(\ell,n)$ is the drawdown in the ℓ-th cell at the end of the n-th time period due to aggregated pumpage and recharge in all other cells (by other users) in the region.

$d_\ell(k_\ell,n)$ is given by:

$$d_\ell(k_\ell,n) = \sum_{j=1}^{m_\ell} \sum_{i=1}^{n} \beta_\ell(k_\ell,j,n-i+1) q_\ell(j,i) - \sum_{i=1}^{n} \left[\beta_\ell(k_\ell,v_\ell,n-i+1) \cdot \sum_{u=1}^{u_\ell} v_\ell(u,i) \right] \quad (6.3)$$

where $\beta_\ell(k_\ell,j,m-i+1)$ is the algebraic technological term relating the drawdown at the k_ℓ-th well to the pumping of one unit of water from the j-th well during the i-th period, and both k_ℓ and j are located at the ℓ-th cell.

The second term on the right of (6.3) stands for the negative drawdown at well k_ℓ caused by the artificial recharge at point v_ℓ.

$D(\ell,n)$ is given by:

$$D(\ell,n) = \sum_{\substack{r=1 \\ r\neq\ell}}^{L} \sum_{i=1}^{n} \gamma(\ell,r,n-i+1) \cdot (q(r,i) - v(r,i)) \quad (6.4)$$

where $\gamma(\ell,r,n-i+1)$ is the algebraic technological term relating the average drawdown at the ℓ-th cell to aggregated pumping of one unit of water at the r-th cell, during the i-th period. $q(r,i)$ is the quantity of water pumped from the r-th cell by the r-th user during the i-th period,

$$q(r,i) = \sum_{k_r=1}^{m_r} q_r(k_r,i) \tag{6.5}$$

for the r-th cell:

$$v(r,i) = \sum_{u=1}^{u_r} v_r(u,i) \tag{6.6}$$

Equation (6.2) contains the products of $q_\ell(k_\ell,n)$ and $q(r,i)$, $r\neq\ell$, i.e. the products of pumping values on the ℓ-th cell area at particular wells and aggregated pumping values on all other cells in the region. The coupling of $q_\ell(k_\ell,n)$ and $v(r,i)$, the aggregated artificial recharge at other cells, is similar.

Two vectors of psuedo-variables $\sigma_1(r,n)$, $\sigma_2(r,n)$ are introduced into equation (6.2). These vectors will uncouple the pumping values on the ℓ-th area wells from all other cells' pumpages and recharges.

Let

$$\left.\begin{aligned} \sigma_1(r,n) &= q(r,n) \\ \sigma_2(r,n) &= v(r,n) \end{aligned}\right\} \quad r=1,\ldots,L;\ n=1,\ldots,T \tag{6.7, 6.8}$$

Then equation (6.2) becomes:

$$Z_\ell = \sum_{n=1}^{T} (1+p)^{-n} \{C_\ell(n) + \sum_{k_\ell=1}^{m_\ell} P_\ell(k_\ell) q_\ell(k_\ell,n) \left[H_\ell(k_\ell) + \sum_{j=1}^{m_\ell} \beta_\ell(k_\ell,j,n-i+1)\right.$$

$$\left. .q_\ell(j,i) - \sum_{i=1}^{n} (\beta_\ell(k_\ell,v_\ell,n-i+1) . \sum_{u=1}^{u_\ell} v_\ell(u,i)) + \sum_{\substack{r=1 \\ r\neq\ell}}^{L} \gamma(\ell,r,n-i+1)\right.$$

$$\left. \left[\sigma_1(r,i) - \sigma_2(r,i))\right]\right\} \tag{6.8}$$

Notice here that the dimension of the vector of psuedo-variables is reduced with respect to the original scheme. The psuedo-variables account for the aggregated activities of each user. The possible 'estimation' by one user of pumpage and recharge planned by others is much more feasible for aggregated operation than it is for a detailed plan applied to each well. Hence, this solution strategy thus provides both a conceptual and methodological advantage.

If the ℓ-th user estimates a set of (L-1) net aggregated pumping values $(\sigma_1(r,i) - \sigma_2(r,i))$ for the L-1 users, then these estimates become the set of pseudo-variables.

The ℓ-th user is interested in maximizing NR_ℓ, subject to such constraints as:

I) Non decreasing water supply

$$\sum_{k_\ell=1}^{m_\ell} q_\ell(k_\ell,n) + QI_\ell(n) \leq \sum_{k_\ell=1}^{m_\ell} q_\ell(k_\ell,n+1) + QI_\ell(n+1) \qquad n=1,\ldots,T-1 \qquad (6.9)$$

II) Drawdowns must not exceed casing and screening designs

$$d_\ell(k_\ell,n) + D(\ell,n) \leq d_{\ell max}(k_\ell) \qquad \begin{matrix} n=1,\ldots,T \\ k_\ell=1,\ldots,m_\ell \end{matrix} \qquad (6.10)$$

III) Pumping capacity must not be exceeded

$$q_\ell(k_\ell,n) \leq Q_{\ell max}(k_\ell) \qquad \begin{matrix} n=1,\ldots,T \\ k_\ell=1,\ldots,m_\ell \end{matrix} \qquad (6.11)$$

IV) Upper limit for imported quantities

$$QI_\ell(n) \leq QI_{\ell max}(n) \qquad n=1,\ldots,T \qquad (6.12)$$

V) Recharge facilities capacity must not be exceeded

$$\sum_{u=1}^{u_\ell} v_\ell(u,n) \leq v_{\ell max} \qquad n=1,\ldots,T \qquad (6.13)$$

$d_{\ell max}(k_\ell)$ is the maximum drawdown allowed for the k_ℓ-th well located at the ℓ-th cell, which must not be exceeded because of casing and screening design.

$Q_{\ell max}(k_\ell)$ is the design upper limit on the quantity of water pumped from the k_ℓ-th well.

$QI_{\ell max}(n)$ is the externally imposed restriction of an upper limit on the quantity of water to be imported into the region for the direct use of the ℓ-th user during the n-th period.

$v_{\ell max}$ is the designed upper limit on the quantity of water to be artificially recharged in the ℓ-th cell recharge facilities.

The regional objective is to enhance the regional net return from water use. As such, the regional optimization problem definition is:

$$\max NR = \sum_{\ell=1}^{L} NR_{\ell} \tag{6.14}$$

Subject to:

I) A lower limit for each user's net benefit

$$NR_{\ell} \geq NR_{\ell min} \qquad \ell=1,\ldots,L \tag{6.15}$$

II) A set of mass balance constraints

$$\left.\begin{aligned} \sigma_1(r,n) - q(r,n) &= 0 \\ \sigma_2(r,n) - v(r,n) &= 0 \end{aligned}\right\} \qquad \begin{matrix} n=1,\ldots,T \\ r=1,\ldots,L \end{matrix} \tag{6.16}$$

III) A set of interference constraints

$$D(\ell,n) \leq D_{\ell max} \qquad \begin{matrix} n=1,\ldots,T \\ \ell=1,\ldots,L \end{matrix} \tag{6.17}$$

IV) Water Balance must be maintained in certain streams

$$\sum_{\ell=1}^{L} \left[v_{\ell}(u,n) + f_{\ell}^{u}(n) \right] \leq B(u,n) \qquad \begin{matrix} n=1,\ldots,T \\ u=1,\ldots,U \end{matrix} \tag{6.18}$$

V) All previous individual user constraints (Equations (6.9) through (6.13)).

$NR_{\ell}min$ is the minimum expected net benefit associated with water use by the ℓ-th user over the planning period.

$D_{\ell max}$ is the upper limit to the drawdown induced by other users activities on the ℓ-th user.

$B(u,n)$ is an upper limit on the quantity of water to be removed from the u-th stream for natural and artificial recharge.

$f_{\ell}^{u}(n)$ is the quantity of water induced from the u-th stream into the ℓ-th cell during the n-th period:

$$f_{\ell}^{u}(n) = \sum_{k_{\ell}=1}^{m_{\ell}} \sum_{i=1}^{n} \phi_{\ell}^{u}(k_{\ell},n-i+1)q_{\ell}(k_{\ell},i) + \sum_{\substack{r=1 \\ r\neq\ell}}^{L} \sum_{i=1}^{n} \psi_{\ell}^{u}(r,n-i+1)\left[q(r,i)-v(r,i)\right] - \sum_{i=1}^{n} \left[\phi_{\ell}^{u}(v_{\ell},n-i+1) \cdot \sum_{u=1}^{u_{\ell}} v_{\ell}(u,i)\right] + I_{\ell}^{u} \tag{6.19}$$

$\phi_{\ell}^{u}(k_{\ell},n-i+1)$ is the quantity of water induced from the u-th stream into the ℓ-th cell during the n-th period due to one unit of pumping at the k_{ℓ}-th well during the i-th period.

$q_{\ell}(k_{\ell},i)$ is the quantity of water pumped from the k_{ℓ}-th well during the i-th period.

$\psi_{\ell}^{u}(r,n-i+1)$ is the quantity of water induced from the u-th stream into the ℓ-th cell during the n-th period due to one unit of pumping at the r-th cell during the i-th period.

$q(r,i) - v(r,i)$ is the net quantity of water pumped from the r-th cell during the i-th period.

I_{ℓ}^{u} is the quantity of water induced from the u-th stream into the ℓ-th cell during one time period with no imposed pumpage and the system in steady state.

The primal solution of the program constituting equations (6.9) through (6.18) provides the quotas for each well and recharge from the stream for each user. The dual solution provides the costs and savings associated with changes in the values of pumpage and recharge. In particular, $q_{\ell}(k_{\ell},n)$ is the quota for the k_{ℓ}-th well of the ℓ-th user for the n-th time period, and $v_{\ell}(u,n)$ is the quota for the quantity of water to be used for artificial recharge at the ℓ-th area from the u-th stream during the n-th time period.

The Lagrangian for the maximum regional return program (equations (6.9) through (6.18) is formed as follows (where NR_ℓ is given by equation (6.1) and (6.2)):

$$
\begin{aligned}
L = & \sum_{\ell=1}^{L} NR_\ell + \sum_{\ell=1}^{L}\sum_{n=1}^{T-1} \mu_\ell^{(1)}(n)\left[\sum_{k_\ell=1}^{m_\ell} q_\ell(k_\ell,n) + QI_\ell(n) - \left(\sum_{k_\ell=1}^{m_\ell} q_\ell(k_\ell,n+1) + QI_\ell(n+1)\right)\right] \\
& + \sum_{\ell=1}^{L}\sum_{k_\ell=1}^{m_\ell}\sum_{n=1}^{T} \mu_\ell^{(2)}(k_\ell,n)\left[d_\ell(k_\ell,n) + D(\ell,n) - d_{\ell max}(k_\ell)\right] \\
& + \sum_{\ell=1}^{L}\sum_{k_\ell=1}^{m_\ell}\sum_{n=1}^{T} \mu_\ell^{(3)}(k_\ell,n)\left[q_\ell(k_\ell,n) - Q_{\ell max}(k_\ell)\right] \\
& + \sum_{\ell=1}^{L}\sum_{n=1}^{T} \mu_\ell^{(4)}(n)\left[QI_\ell(n) - QI_{\ell max}(n)\right] \\
& + \sum_{\ell=1}^{L}\sum_{n=1}^{T} \mu_\ell^{(5)}(n)\left[\sum_{u=1}^{U} V_\ell(u,n) - V_{\ell max}\right] \\
& + \sum_{\ell=1}^{L}\sum_{k_\ell=1}^{m_\ell}\sum_{n=1}^{T} \mu_\ell^{(6)}(k_\ell,n)\left[d_\ell(k_\ell n) - \left\{\sum_{j=1}^{m_\ell}\sum_{i=1}^{n} \beta_\ell(k_\ell,j,n-i+1)\, q_\ell(j,i)\right.\right. \\
& \qquad \left.\left. \sum_{i=1}^{n}\left[\beta_\ell(k_\ell, v_\ell, n-i+1) \sum_{u=1}^{u_\ell} v_\ell(u,i)\right]\right\}\right] \\
& + \sum_{\ell=1}^{L}\sum_{n=1}^{T} \mu_\ell^{(7)}(n)\left[D(\ell,n) - \sum_{r=1}^{L}\sum_{i=1}^{n} \gamma(\ell,r,n-i+1)\,(q(r,i) - v(r,i))\right] \\
& \qquad\qquad r\neq\ell \\
& + \sum_{\ell=1}^{L}\sum_{n=1}^{T} \mu_\ell^{(8)}(n)\left[q(\ell,n) - \sum_{k_\ell=1}^{m_\ell} q_\ell(k_\ell,n)\right] \\
& + \sum_{\ell=1}^{L}\sum_{n=1}^{T} \mu_\ell^{(9)}(n)\left[v(\ell,n) - \sum_{u=1}^{u_\ell} v_\ell(u,n)\right] \\
& + \sum_{\ell=1}^{L} \mu_\ell^{(10)}\left[NR_{\ell min} - NR_\ell\right] \\
& + \sum_{\ell=1}^{L}\sum_{n=1}^{T} \mu_\ell^{(11)}(n)\left[D(\ell,n) - D_{\ell max}\right]
\end{aligned}
$$

$$+ \sum_{u=1}^{U} \sum_{n=1}^{T} \mu^{(12)}(u,n) \left[\sum_{\ell=1}^{L} \left(v_\ell(u,n) + f_\ell^u(n) - B(u,n) \right) \right]$$

$$+ \sum_{\ell=1}^{L} \sum_{u=1}^{u_\ell} \sum_{n=1}^{T} \mu_\ell^{(13)}(u,n) \left[f_\ell^u(n) - \left\{ \sum_{k_\ell=1}^{m_\ell} \sum_{i=1}^{n} \phi_\ell^u(k_\ell, n-i+1)\, q_\ell(k_\ell,i) \right.\right.$$

$$- \sum_{i=1}^{n} \phi_\ell^u(v_\ell, n-i+1) \cdot \sum_{u=1}^{u_\ell} v_\ell(u,i)$$

$$+ \sum_{\substack{r=1 \\ r\neq\ell}}^{L} \sum_{i=1}^{n} \psi_\ell^u(r, n-i+1)(q(r,i)-v(r,i))$$

$$\left.\left. + I_\ell^u \right\} \right]$$

$$+ \sum_{\ell=1}^{L} \sum_{n=1}^{T} \lambda_\ell^{(1)}(n) \left[\sigma_1(\ell,n) - q(\ell,n) \right]$$

$$+ \sum_{\ell=1}^{L} \sum_{n=1}^{T} \lambda_\ell^{(2)}(n) \left[\sigma_2(\ell,n) - V(\ell,n) \right] \qquad \ldots\ldots\ldots\ldots (6.20)$$

Applying the multilevel decomposition approach, the Lagrangian L is decomposable into L independent subsystems where all psuedo-variables are assumed to be known parameters at the first level (i.e., to the users) quadratic program optimization:

$$L = \sum_{\ell=1}^{L} L_\ell \tag{6.21}$$

And L_ℓ is the Lagrangian for the ℓ-th subsystem.

The decision variables of subsystem ℓ at the first level optimization are

$$q_\ell(k_\ell,n)'s,\ v_\ell(u,n)'s,\ QI_\ell(n)'s$$

$$D(\ell,n)'s \text{ and } \mu_\ell^{(p)},\ p = 1,\ldots,6,8,9.$$

The global optimum of the problem is guaranteed when the quadratic functions are convex.

The decision variables for the second level coordination are

$$\sigma_1(\ell,n)'s,\ \sigma_2(\ell,n)'s,\ \mu_\ell^{(P)},\ P=7,10,11,12,13,$$

$$\text{and } \lambda_\ell^{(1)}(n),\ \lambda_\ell^{(2)}(n)$$

Applying some of the Kuhn-Tucker necessary conditions (for stationarity) at the second level optimization yields:

$$\frac{\partial L}{\partial \sigma_1(r,i)} = 0 = -\sum_{n=i}^{T}\left[(1+\rho)^{-n}\sum_{\substack{\ell=1\\ \ell\neq r}}^{L}\sum_{k_\ell=1}^{m_\ell} P_\ell(k_\ell)q_\ell(k_\ell,n)\ .\ \gamma(\ell,r,n-i+1)\right] + \lambda_{r(i)}^{(1)} \tag{6.22}$$

$$\frac{\partial L}{\partial \sigma_2(r,i)} = 0 = \sum_{n=i}^{T}(1+\rho)^{-n}\sum_{\substack{\ell=1\\ \ell\neq r}}^{L}\sum_{k_\ell=1}^{m_\ell} P_\ell(k_\ell)q_\ell(k_\ell,n)\ .\ \gamma(\ell,r,n-i+1) + \lambda_r^{(2)}(i) \tag{6.23}$$

$$\frac{\partial L}{\partial \lambda_\ell^{(1)}(n)} = 0 = \sigma_1(\ell,n) - q(\ell,n) \tag{6.24}$$

$$\frac{\partial L}{\partial \lambda_{\ell}}(2)_{(n)} = 0 = \sigma_2(\ell,n) - v(\ell,n) \tag{6.25}$$

which results in:

$$\lambda_r^{(1)}(i) = -\lambda_r^{(2)}(i) = \sum_{n=i}^{T} (1+\rho)^{-n} \sum_{\ell=1}^{L} \overline{P}_{\ell} \cdot \gamma(\ell,r,n-i+1) \cdot q(r,n) \tag{6.26}$$

where

$$\overline{P_{\ell}} = \left[\sum_{n=1}^{T} \sum_{k_{\ell}=1}^{m_{\ell}} P_{\ell}(k_{\ell})\, q_{\ell}(k_{\ell},n)\right] \Big/ \left[\sum_{n=1}^{T} \sum_{k_{\ell}=1}^{m_{\ell}} q_{\ell}(k_{\ell},n)\right] \tag{6.27}$$

At the second level of the hierarchy equations (6.24), (6.25), (6.26) are determined by inserting the 'optimal' values of q(r,n) and v(r,n) produced by the first level optimization. An iterative procedure between the first level and the second level is initiated. The first level supplies the second level with q's and v's and the second level supplies the first level with $c\sigma^{(1)}$'s, $\sigma^{(2)}$'s and λ's.

At this point the advantage of the above formulation in comparison with the original study can be appreciated. The iterative procedure originally required the pumping values as well as corresponding the psuedo-variables to originate between the two levels for each well. Using the concepts developed previously in our study, only aggregated activities (pumpage and recharge) and their corresponding psuedo-variables (σ_1,σ_2) are iterated. The dimensionality of the procedure is obviously reduced and convergence is expected to be achieved more rapidly.

The Lagrange multipliers $\lambda^{(1)}$ given by (6.26) are the dual variables corresponding to the constraints: $\sigma_1(\ell,n) - q(\ell,n) = 0$. These represent a cost per unit excess of over-pumping the quota by each user. Notice that in contrast to the original study's scheme, the quota system in the above formulation corresponds to the aggregated pumpage by each user. (Originally quotas were determined for each particular well. This raises sensitivity problems due to the possibility of mechanical failures in well equipment (or such other difficulties)

which might not allow the user to operate his well system exactly as the quota system would impose.)

The Lagrange multipliers $\lambda^{(2)}$ are the dual variables corresponding to the constraints $\sigma_2(\ell,n) - v(\ell,n) = 0$. These represent a saving per unit excess of over-recharge or a cost per unit of under recharge, relative to the recharge quota.

A more detailed discussion on the quota system and the different assumptions is given in the original paper.

6.3 Tax Computation

In the following, a modification of the taxation scheme suggested by Maddock and Haimes is developed. The basic assumption used is that under a feasible tax scheme applied to groundwater pumping, users may cooperate for a tax collection system on an aggregated basis. In other words, each user desires to operate his own wells and recharge facilities according to his own considerations given the aggregated quotas imposed on him. He may reject any attempt to impose a pumping plan for his wells not in correspondence with his own planned operations.

Let $\Delta q(\ell,i)$, $\Delta v(\ell,i)$ (if positive) be the respective expressions for the ℓ-th user pumping and recharging more than his quotas. Then

$$C_o(i) = \sum_{\ell=1}^{L} \lambda_\ell^{(1)}(i) \cdot \left[\Delta q(\ell,i) - \Delta v(\ell,i)\right] \qquad (6.28)$$

is the cost (saving if negative) of additional energy that all users have to expend over the remainder of the planning periods to produce their quotas. Expression (6.28) stands for the total tax collected from all users at year i.

Let

$$C_I(\ell,i) = \sum_{\substack{r=1 \\ r\neq\ell}}^{L} \sum_{n=i}^{T} (1+\rho)^{-n} \bar{P}_\ell \cdot \gamma(\ell,r,n-i+1) q(r,n) \left[\Delta q(r,i) - \Delta v(r,i)\right] \qquad (6.29)$$

Denote the total cost to the ℓ-th user due to over activities by other users, then the total cost to all users is

$$C_I^T(i) = \sum_{\ell=1}^{L} C_I(\ell,i) \tag{6.30}$$

Since equation (6.30) is equivalent to (6.28):

$$C_o(i) - C_I^T(i) = \sum_{\ell=1}^{L}\sum_{n=i}^{T}(1+\rho)^{-n}\sum_{\substack{r=1\\r\neq\ell}}^{L}\overline{P}_r \cdot \gamma(r,\ell,n-i+1)q(\ell,n)\left[\Delta q(\ell,i) - \Delta v(\ell,i)\right] - \sum_{\ell=1}^{L}\sum_{\substack{r=1\\r\neq\ell}}^{L}\sum_{n=i}^{T}(1+\rho)^{-n} P_\ell \gamma(\ell,r,n-i-1)q(r,n) \cdot \left[\Delta q(r,i) - \Delta^V(r,i)\right] = 0 \tag{6.31}$$

The ℓ-th user is assessed the tax

$$T_x(\ell,i) = \sum_{n=i}^{T}(1+\rho)^{-n}\left\{\sum_{\substack{r=1\\r\neq\ell}}^{L}\overline{P}_r\,\gamma(r,\ell,n-i+1)q(\ell,n)\left[\Delta q(\ell,i) - \Delta v(\ell,i)\right] - \sum_{\substack{r=1\\r\neq\ell}}^{L}\overline{P}_\ell\,\gamma(\ell,r,n-i+1)\cdot q(r,n)\left[\Delta q(r,i) - \Delta v(r,i)\right]\right. \tag{6.32}$$

The net collected tax for the i-th time period is zero:

$$\sum_{\ell=1}^{L} T_x(\ell,i) = 0 \tag{6.33}$$

6.4 Summary and Conclusions

The tax-quota scheme developed by Maddock and Haimes for a simple, isolated aquifer system has been modified in this study for a more general complex groundwater system. The application of the concept of decomposed response functions to the problem formulation makes it possible to account for a vast range of variables affecting decisions.

In our development two aspects of usefulness of the decomposed response functions are illustrated:

I) Simplification of the mathematical formulation and the solution strategy;

II) Extension of the model to handle more of those items affected by the activities considered (e.g., artificial recharge options and stream network response).

In the context of our study, the modified tax-quota system model may be viewed as an illustration of the application of a management scheme to a region in the hopes of initiating an implementation of a management mechanism. The regional performance criterion under the porposed mechanism is expected to considerably improve results obtained from the basic non-management mechanism structure.

In the final stage of our study we intend to formulate the problem as a multi-objective function coordination scheme. The tax-quota system in its modified form may then be further improved, perhaps by adding more management mechanisms to utilize the multi-objective function analysis and application.

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A P P E N D I X A

THE GREEN'S FUNCTION

A.1 THE CONCEPT OF A GREEN'S FUNCTION

One method (of many) developed for solving boundary value problems requires the construction of an auxiliary function known as a Green's Function. To show how such functions arise, we will first solve, by fairly elementary methods, a typical one dimensional boundary value problem. Consider the following differential equation for the unknown variable U [Roach, 1970].

$$\frac{\partial^2 U}{\partial x^2} + k^2 U = - f(x) \qquad 0 \leq x \leq \ell \tag{A.1}$$

with boundary conditions $U(0) = U(\ell) = 0$ (A.2)

Assume the solution:

$$U(x) = A(x)\cos kx + B(x) \sin kx \tag{A.3}$$

Differentiate (3) twice with respect to x and assume

$$A'\cos kx + B'\sin kx = 0 \tag{A.4}$$

Then (3) constitutes a solution provided that

$$-KA'\sin kx + KB' \cos \quad = - f(x) \tag{A.5}$$

Equations (A-4) and (A-5) are two linear algebraic equations for $A'(x)$ and $B'(x)$. We find

$$A'(x) = \frac{f(x)\sin KX}{K} \qquad B'(x) = \frac{-f(x)\cos kx}{K} \tag{A.6}$$

And conclude:

$$U(x) = \frac{\cos kx}{k} \int_{c_1}^{x} f(y)\sin ky \, dy - \frac{\sin kx}{k} \int_{c_2}^{x} f(y) \cos ky \, dy \tag{A.7}$$

C_1 and C_2 are constants to be determined by the boundary condition (A-2).

$$U(o) = o \rightarrow \int_{c_1}^{o} f(y) \sin ky \, dy = 0 \quad \text{for } f(y) \text{ arbitrary}$$

$$\rightarrow C_1 = 0 \tag{A.8}$$

$$U(\ell) = 0 = \frac{\cos k\ell}{k} \int_{o}^{\ell} f(y) \sin ky \, dy - \frac{\sin k\ell}{k} \int_{c_2}^{\ell} f(y) \cos ky \, dy$$

$$= \frac{\cos k\ell}{k} \int_{o}^{\ell} f(y) \sin ky \, dy - \frac{\sin k\ell}{k} \int_{o}^{\ell} f(y) \cos ky \, dy$$

$$- \frac{\sin k\ell}{k} \int_{c_2}^{o} f(y) \cos ky \, dy$$

$$= \frac{1}{k} \int_{o}^{\ell} f(y) \left[\cos k\ell \sin ky - \sin k\ell \cos ky\right] dy -$$

$$\frac{\sin k\ell}{k} \int_{c_2}^{o} f(y) \cos ky \, dy$$

$$= \frac{1}{k} \int_{o}^{\ell} f(y) \sin k(y-\ell) dy - \frac{\sin k\ell}{k} \int_{c_2}^{o} f(y) \cos ky \, dy = 0 \tag{A.9}$$

$$\rightarrow \int_{c_2}^{o} f(y) \cos ky \, dy = \frac{1}{\sin k\ell} \int_{o}^{\ell} f(y) \sin k(y-\ell) \, dy \tag{A.10}$$

The solution (A 7) can now be written

$$U(X) = \frac{\cos kx}{k} \int_{o}^{x} f(y) \sin ky \, dy - \frac{\sin kx}{k} \left[\frac{1}{\sin k\ell} \int_{o}^{\ell} f(y) \sin k(y-\ell) dy\right.$$

$$\left. + \int_{o}^{x} f(y) \cos ky \, dy\right]$$

$$= \frac{1}{k} \int_0^x f(y) \sin k(y-x)\, dy - \frac{\sin kx}{k \sin k\ell} \int_0^{\ell} f(y) \sin k(y-\ell)\, dy$$

$$= \int_0^x f(y) \frac{\sin ky \sin k(\ell-x)}{k \sin k\ell}\, dy + \int_x^{\ell} f(y) \frac{\sin kx \sin k(\ell-y)}{k \sin k\ell}\, dy \qquad (A.11)$$

$$U(x) = \int_0^{\ell} f(y)\, G(x,y)\, dx \qquad (A.12)$$

where G(x,y) is defined as

$$G(x,y) = \frac{\sin ky \sin k(\ell-x)}{k \sin k\ell} \qquad 0 \le y \le x$$

$$= \frac{\sin kx \sin k(\ell-y)}{k \sin k\ell} \qquad x \le y \le \ell \qquad (A.13)$$

G(x,y) is the Green's Function for equation (A-1) and boundary condition (A 2). More details on existence conditions are in Roach [1970]. G(x,y) is independent of the forcing term f(x), and depends only upon the particular differential equation and boundary conditions.

Properties of G(x,y):

1) It satisfies the homogeneous form of the given differential equation:

$$G'' + k^2 G = 0 \qquad (A.14)$$

In the intervals $0 \le y < x \qquad x < y \le \ell$

We shall later discuss its behavior at y=x.

2) G(x,y) is continuous at y=x.

3) The derivative of G w.r.t. Y is discontinuous at y=x.

4) G(x,y) satisfies the boundary conditions: $G(x,o) = G(x,\ell) = 0$.

5) G(x,y) is symmetric, G(x,y) = G(y,x).

A general form of the problem is $Lu = f$ where L is a linear operator.

We define the inverse operator L^{-1} where $L^{-1} L = I$, the identity operator. If L is a linear, ordinary differential operator, L^{-1} (if it exists) takes the form of an integral operator, the kernel of which is the Green's Function $G(x,y)$ for the operator L.

$$L^{-1}u = \int K(x,t)\, u(t)\, dt \tag{A.15}$$

and

$$u(x) = Iu = LL^{-1}u = L \int K(x,t)u(t)\, dt = \int LK(x,t)\, u(t)\, dt \tag{A.16}$$

The kernel of this integral is:

$$LK(x,t) = g(x,t)$$

and (A.17)

$$u(x) = \int g(x,t)u(t)dt$$

If this result is true for all continuous functions $u(t)$, it follows that $g(x,t)$ must be zero whenever $x \neq t$. When $x=t$, the integral on the right must reduce identically to $u(x)$.

We define $g(x,t) = \delta(t-x)$ where δ is the Dirac Function with the properties:

$$u(x) = \int \delta(t-x)u(t)dt$$

$$\delta(x) = 0 \quad \text{if } x \neq 0 \tag{A.18}$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

And for every continuous function $\phi(x)$,

$$\int_{\infty}^{\infty} \delta(x)\ \phi(x)\ dx = \phi(0) \tag{A.19}$$

We may now conclude the discussion regarding the Green's Function Property for the one-dimensional case. The first property is reformulated for the interval $0 \leq y \leq \ell$ as:

$$G'' + k^2 G = \delta(x-y) \tag{A.20}$$

A.2 The Green's Function for the Aquifer Model

The generalized model for a groundwater system with finite boundaries, nonhomogeneous transmisivity, and non-homogeneous storage coefficients, has the differential equation: [Pinder and Bredehoft, 1968].

$$\frac{\partial}{\partial x'}\left[T(x')\frac{\partial}{\partial x'} d(x',t)\right] = s(x')\frac{\partial}{\partial t} d(x',t) + \sum_{k=1}^{m} Q(k,t)\,\delta(x'-x_k) \tag{A.21}$$

where x' is the point (x,y), T(x') the transmissivity coefficient and s(x') the storage coefficient. Q(k,t) is the pumpage at well k, and the variable d(x',t) denotes the drawdown at x' in time t.

The initial conditions are $d(x',t) = 0.$ (A.22)

The boundary conditions are no flow boundaries:

$$\frac{\partial d(\lambda)}{\partial n} = 0 \tag{A.23}$$

where λ is a parameter indicating that $\frac{\partial d}{\partial n}$ is evaluated on the boundary, and the n is normal direction to the boundary.

Let the solution for the drawdown distribution be the integral: [Maddock, 1972]

$$d(x',t) = \sum_{j=1}^{m} \int_{o}^{t} \alpha(x',x_j',t-\tau)Q(x_j,\tau)\,d\tau \tag{A.24}$$

where x' is the point of observation, x_j''s are well locations and $\alpha(x',x_j',t)$ the presently unknown kernel that is a function of x',x_j', and t. Notice that:

$$\sum_{j=1}^{n} Q(x_j,t)\,\delta(x'-x_j') = \sum_{j=1}^{m} \int_{o}^{t} Q(x_j',\tau)\delta(t-\tau)\delta(x'-x_j')\,d\tau \tag{A.25}$$

Substituting (A-24) and (A-25) into (A-21)

$$\frac{\partial}{\partial x'}\left[T(x')\frac{\partial}{\partial x'}\left(\sum_{j=1}^{m}\int_{o}^{t}\alpha(x',x_j',\tau)Q(x_j',t-\tau)d\tau\right)\right] - s(x')\frac{\partial}{\partial t}\left[\sum_{j=1}^{m}\int_{o}^{t}\alpha(x',x_j',\tau)Q(x_j',t-\tau)d\tau\right] + \sum_{j=1}^{m}\int_{o}^{t}Q(x_j',t-\tau)\delta(\tau)\delta(x'-x_j')d\tau = 0$$

$$\rightarrow \sum_{j=1}^{m}\int_{o}^{t}\left\{\frac{\partial}{\partial x'}\left[T(x')\frac{\partial}{\partial x'}\alpha(x',x_j',\tau) - s(x')\frac{\partial}{\partial t}\alpha(x',x_j',\tau) + \delta(\tau)\delta(x'-x_j')\right\}Q(x_j',t-\tau)dt = 0 \qquad (A.26)$$

If $\alpha(x',x_j',t) = G(x',x_j',t)$, The Green's Function for equation (A-21), then according to the first property of the Green's function the following equation is satisfied:

$$\frac{\partial}{\partial x'}\left[T(x')\frac{\partial}{\partial x'}G(x',x_j',t)\right] - s(x')\frac{\partial G}{\partial t}(x',x_j',t) = -\delta(t)\delta(x'-x_j') \qquad (A.27)$$

and from the fourth property, the initial boundary conditions must hold:

$$G(x',x_j',0) = 0 \qquad (A.28)$$

that is,

$$G(x',x_j',t-\tau) = 0 \qquad t < \tau \qquad (A.29)$$

If the time horizon is designed to comprise n equal seasons, the time history of pumping from the k-th well up to and including the n-th season is Q(k,n), where:

$$Q(k,n) = \sum_{i=1}^{n} q(k,i)\{u[t-(i-1)\eta]-u[t-i\eta]\} \qquad (A.30)$$

and $q(k,i)$ is the discharge from the k-th well during the i-th season. The seasons are of duration η, and $u(t-i\eta)$ is the unit step function, defined as

$$\begin{aligned} u(t-i\eta) &= 0 \qquad t \leq i\eta \\ &= 1 \qquad t > i\eta \end{aligned} \tag{A.31}$$

If in equation (A-24) the time integral is replaced by a sum over n periods of constant duration η, then for observation point k we get

$$d(k,n_\eta) = \sum_{j=1}^{m} \sum_{i=0}^{n_\eta} \alpha(k,x_j', n_\eta - i)Q(x_j',i) \tag{A.32}$$

Let η= one unit of time. Substituting equation (A-30) for an arbitrary time period n into (A-32), letting j be the index denoting pumpage from well at location x_j', and defining

$$\begin{aligned} \beta(k,j,i) &= \alpha(k,j,i) \qquad i=1 \\ &= \alpha(k,j,i) - \alpha(k,j,i-1) \quad i > 1 \end{aligned} \tag{A.33}$$

Equation (A-32) then becomes:

$$d(k,n) = \sum_{j=1}^{m} \sum_{i=1}^{n} \beta(k,j,n-i+1)q(j,i) \tag{A.34}$$

Equation (A-34) is an algebraic technological function (A.T.F.) that determines the drawdown at the k-th well at the end of the n-th season due to the pumping of m wells.

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A P P E N D I X B

Computer Program of a Multicell Model
For the Fairfield New-Baltimore Area

```
      JOB,LPT,OVRR MULTICELL
      REMOVE,MODEL,PERMANENT
      COMPILE,MODEL,PERMANENT,DOUBLE
C  MULTICELL MODEL FOR THE FAIRFIELD-NEW BALTIMOR AREA
C
C  MAKES USE OF THE EXPLICIT METHOD TO SOLVE THE SET
C  OF BALANCE EQUATIONS OF THE CELLS.
C
C  Z(I) IS THE I-TH EXPRESSION FOR THE FLOW BETWEEN CERTAIN ADJACENT
C   CELLS.
C
C  H(I,M+1) IS THE BALANCE EXPRESSION OF WATER HEAD AT THE I-TH CELL
C  AT THE BEGINING OF THE M+1 TIME PERIOD.
C
C  CELLS 1,3,9-CONSTANT HEAD  ASSUMED FOR ADJACENT AQUIFERS.
C
C  CELLS 10,11,12-CONSTANT HEAD REPRESENTING STREAM REACHES
C
C  CELLS 2,4,5,6,7-REPRESENT SECTIONS OF THE AQUIFER
C   WITH HOMOGENEOUS STORATIVITY AND CONDUCTIVITY.
C
C
        DIMENSION Q(7,96),A(12),S(12),W(12),L(12),K(12),E(12),F(12),
     6   H(12,96),R(12),U(12),V(12),Z(12)
   10 FORMAT(7F8.2)
   20 FORMAT(F10.2,I10,I10)
   25 FORMAT(1H,6X,12(I2,6X))
   30 FORMAT(12(F5.1))
   35 FORMAT(1H,5X,7(I2,8X))
   40 FORMAT(6F10.3)
   45 FORMAT(1H,2X,$I=$,6X,$A(I)=$,6X,$S(I)=$,6X,$W(I)=$,6X,$L(I)=$,6X,
     6   $K(I)=$,6X,$E(I)=$,6X,$F(I)=$//)
   50 FORMAT(1H,2X,I2,2X,7F10.2)
   60 FORMAT(1H,2X,$DELT=$,F5.1,2X,$N=$,I3,2X,$J=$,I3)
   65 FORMAT(1H0,20X,$INITIAL HEADS$)
   70 FORMAT(1H,2X,12(F8.1))
   75 FORMAT(1H0,20X,$PUMPING VALUES$)
   80  FORMAT(1H,2X,6F10.5)
  401 FORMAT(1H0,30X,$FLOW VALUES$)
  402 FORMAT(1H0,30X,$HEAD VALUES$)
  405 FORMAT(24X,12(I2,6X))
  410 FORMAT(1H,20X,12(F6.1))
      REAL L,K
C
C
C  DELT(DAYS)  =THE TIME STEP
C  N(INTEGER)  =# OF TIME STEPS IN A RUN
C
C  J           =-1 READ PUMPING DATA Q FOR EACH YEAR
C              =0 READ Q CONSTANT FOR THE WHOLE RUN
C              =1 READ Q FOR EACH TIME STEP
C
      READ 20,DELT,N,J
      PRINT 60,DELT,N,J
```

```
      C
      C
      C  A(I)[FT**2/10**6] =AREA OF CELL I
      C  S(I)[DIMENSIONLESS]=STORATIVITY AT I-TH CELL
      C
      C  THE VALUES ASSIGNED TO N,L,K,E,F,/ IN THE FOLLOWING CORRESPOND
      C   TO RELATIONS BETWEEN CELLS:
      C
      C    I    CELLS
      C    1     2←1
      C    2     2←4
      C    3     4←3
      C    4     4←10
      C    5     6←5
      C    6     5←7
      C    7     7←6
      C    8     7←8
      C    9     6←9
      C   10     5←10
      C   11     5←11
      C   12     7←12
      C
      C
      C  W(I)[FT]     =LENGTH OF THE PERPENDICULAR SECTOR ASSOCIATED
      C                WITH THE SEGMENT BETWEEN CELLS
      C
      C  L(I)[FT]     =DISTANCE BETWEEN CENTERS OF CELLS
      C
      C  K(I)[FT/DAY]=HYDRAULIC CONDUCTIVITY AVERAGED BETWEEN CELLS
      C
      C  E(I)[FT]     =EFFECTIVE AQUIFER DEPTH AVERAGED BETWEEN CELLS
      C
      C  F(I)[FT]     =ELEVATION AT THE TOP OF THE AQUIFER AVERAGED BETWEEN
      C                CELLS
      C
      C
                  PRINT 45
            DO 100 I=1,12
               READ 10,A(I),S(I),W(I),L(I),K(I),E(I),F(I)
              PRINT 50,I,A(I),S(I),W(I),L(I),K(I),E(I),F(I)
        100 CONTINUE
      C
      C  READ INITIAL HEAD DISTRIBUTION
      C
                READ 30,(H(I,1),I=1,12)
                   PRINT 65
                  PRINT 25,(I,I=1,12)
                 PRINT 70,(H(I,1),I=1,12)
                      PRINT 75
                  PRINT 35,(I,I=2,7)
               IF(J) 215,200,220
      C
      C  J=0 READ PUMPAGE DATA Q[FT**3/DAY/10**6]  AT CELLS
      C       CONSTANT FOR THE WHOLE RUN
```

```
C
  200  READ 40,(Q(I,N),I=2,7)
      PRINT 30,(Q(I,N),I=2,7)
            DO 210 N=1,N
                  DO 205 I=2,7
                Q(I,M)=Q(I,N)*1000
  205       CONTINUE
  210  CONTINUE
        GOTO 240
C
C  J=-1 READ PUMPAGE DATA AT CELLS FOR EACH YEAR
C
  215  LT=365./DELT
        NN=N-LT+1
          DO 219 MM=1,NN,LT
          MMM=MM+LT-1
                  READ 40,(Q(I,MMM),I=2,7)
                PRINT 80,(Q(I,MMM),I=2,7)
                      DO 218 M=MM,MMM
                         DO 217 I=2,7
                              Q(I,M)=1000*Q(I,MMM)
  217                         CONTINUE
  218                   CONTINUE
  219            CONTINUE
          GOTO 240
C
C  J=1  READ PUMPAGE DATA FOR EACH TIME STEP
C
  220 DO 230 M=1,N
       READ 40,(Q(I,M),I=2,7)
      PRINT 80,(Q(I,M),I=2,7)
              DO 225 I=2,7
           Q(I,M)=1000*Q(I,M)
  225       CONTINUE
  230 CONTINUE
  240 DO 300 I=1,12
         R(I)=W(I)*K(I)*(F(I)-F(I))/L(I)
         U(I)=W(I)*K(I)/(2.*L(I))
         V(I)=1000000*A(I)*S(I)/DELT
       DO 250 M=2,N
          H(I,M)=H(I,1)
  250  CONTINUE
  300 CONTINUE
C
C  EXPLICIT CALCULATION OF CELLS HEAD WITH TIME
C
              PRINT 401
      PRINT 405,(I,I=1,12)
      DO 400 M=1,N
C
C  FLOW BETWEEN CELLS (SEE TABLE OF ASSIGNMENTS)
C
      Z(1)=R(1)*(H(1,M)-H(2,M))+U(1)*(H(1,M)**2.-H(2,M)**2.)
      Z(2)=R(2)*(H(4,M)-H(2,M))+U(2)*(H(4,M)**2.-H(2,M)**2.)
```

```
      Z(3)=R(3)*(H(3,M)-H(4,M))+U(3)*(H(3,M)**2.-H(4,M)**2.)
      Z(4)=R(4)*(H(10,M)-H(4,M))+U(4)*(H(10,M)**2.-H(4,M)**2.)
      Z(5)=R(5)*(H(5,M)-H(6,M))+U(5)*(H(5,M)**2.-H(6,M)**2.)
      Z(6)=R(6)*(H(7,M)-H(5,M))+U(6)*(H(7,M)**2.-H(5,M)**2.)
      Z(7)=R(7)*(H(6,M)-H(7,M))+U(7)*(H(6,M)**2.-H(7,M)**2.)
      Z(8)=R(8)*(H(8,M)-H(7,M))+U(8)*(H(8,M)**2.-H(7,M)**2.)
      Z(9)=R(9)*(H(9,M)-H(6,M))+U(9)*(H(9,M)**2.-H(6,M)**2.)
      Z(10)=R(10)*(H(10,M)-H(5,M))+U(10)*(H(10,M)**2.-H(5,M)**2.)
      Z(11)=R(11)*(H(11,M)-H(5,M))+U(11)*(H(11,M)**2.-H(5,M)**2.)
      Z(12)=R(12)*(H(12,M)-H(7,M))+U(12)*(H(12,M)**2.-H(7,M)**2.)
C
C  HEAD AT CELLS FOR ALL TIME STEPS
C
      H(2,M+1)=(Z(1)+Z(2)+Q(2,M))/V(2)+H(2,M)
      H(4,M+1)=(-Z(2)+Z(3)+Z(4)+Q(4,M))/V(4)+H(4,M)
      H(5,M+1)=(Z(10)-Z(5)+Z(6)+Z(11)+Q(5,M))/V(5)+H(5,M)
      H(6,M+1)=(Z(5)-Z(7)+Z(9)+Q(6,M))/V(6)+H(6,M)
      H(7,M+1)=(-Z(6)+Z(7)+Z(8)+Z(12)+Q(7,M))/V(7)+H(7,M)
                    DO 350 I=1,12
           Z(I)=Z(I)/1000.
  350         CONTINUE
           PRINT 410,(Z(I),I=1,12)
  400 CONTINUE
C
              PRINT 402
      PRINT 405,(I,I=1,12)
      DO 500 M=1,N
       PRINT 410,(H(I,M),I=1,12)
  500 CONTINUE
      PRINT 405,(I,I=1,12)
        STOP
      END
END COMPILATION
JOB,OFF
```